Quiz 2: Gibbs Distribution

1. Consider two electrons in a two-level "atom" in equilibrium at temperature T. The level separation is ε.

a) Evaluate the partition function of this system and its free energy.

b) Find the entropy and the energy of the system in the limits $T \gg \varepsilon$ and $T \ll \varepsilon$.

Hint: $\log(1+x) \approx x$, when $x \ll 1$.

Suppose the system is now subject to a magnetic field H. The energy of a spin-up electron is $-\mu_B H$ and of a spin-down electron is $\mu_B H$.

c) Evaluate the partition function of this system and its free energy.

d) Evaluate the linear magnetic susceptibility $\chi = -\left(\partial^2 F/\partial H^2\right)_{H=0}$ in the limits $T \ll \varepsilon$ and $T \ll \varepsilon$.

Hint: $\cosh x \approx 1 + x^2/2$, when $x \ll 1$.

Bonus question

Suppose that we are now dealing with a large number of two-level systems and the level spacing is distributed with the probability distribution function $P(\varepsilon) = \Delta^{-1}\exp(-\varepsilon/\Delta)$.

e) Evaluate approximately the mean value of the magnetic susceptibility $\overline{\chi} = \int_0^\infty \chi(\varepsilon)\, P(\varepsilon)\, d\varepsilon$ in the limit $T \ll \Delta$.

Solution

a)

$$Z = 1 + 4\exp(-\varepsilon/T) + \exp(-2\varepsilon/T)$$

$$F = -T\log Z = -T\log\left[1 + \exp(-2\varepsilon/T) + 4\exp(-\varepsilon/T)\right]$$

b)

$$F \approx \begin{array}{ll} -T\left(\exp\left(-2\varepsilon/T\right)+4\exp\left(-\varepsilon/T\right)\right) & T \ll \varepsilon \\ -T\log 6+\varepsilon & T \gg \varepsilon \end{array}$$

$$S = -\frac{\partial F}{\partial T} \approx \begin{array}{ll} 4\left(\varepsilon/T\right)\exp\left(-\varepsilon/T\right) & T \ll \varepsilon \\ \log 6 & T \gg \varepsilon \end{array}$$

$$E = F + TS \approx \begin{array}{ll} 4\varepsilon\exp\left(-\varepsilon/T\right) & T \ll \varepsilon \\ \varepsilon & T \gg \varepsilon \end{array}$$

c)

$$Z = 1 + 2\exp\left(-\varepsilon/T\right) + \exp\left[\left(-\varepsilon+2\mu_B H\right)/T\right] + \exp\left[\left(-\varepsilon-2\mu_B H\right)/T\right] + \exp\left(-2\varepsilon/T\right)$$

$$= 1 + 2\exp\left(-\varepsilon/T\right)\left[1+\cosh\left(2\mu_B H/T\right)\right] + \exp\left(-2\varepsilon/T\right)$$

$$F = -T\log Z = -T\log\left\{1 + 2\exp\left(-\varepsilon/T\right)\left[1+\cosh\left(2\mu_B H/T\right)\right] + \exp\left(-2\varepsilon/T\right)\right\}$$

d)

$$F \approx -T\log\left\{1 + 2\exp\left(-\varepsilon/T\right)\left[2+2\left(\mu_B H/T\right)^2\right] + \exp\left(-2\varepsilon/T\right)\right\}$$

$$\approx \begin{array}{ll} -2T\exp\left(-\varepsilon/T\right)\left[2+2\left(\mu_B H/T\right)^2\right] & T \ll \varepsilon \\ -T\log\left\{2+2\left[2+2\left(\mu_B H/T\right)^2\right]\right\} & T \gg \varepsilon \end{array}$$

$$\approx \begin{array}{ll} -4T\exp\left(-\varepsilon/T\right)\left[1+\left(\mu_B H/T\right)^2\right] & T \ll \varepsilon \\ -T\log 6 - 2T\left[\left(\mu_B H/T\right)^2\right]/3 & T \gg \varepsilon \end{array}$$

$$\chi = -\left(\partial^2 F/\partial H^2\right)_{H=0} = \mu_B^2 \begin{array}{ll} 8T^{-1}\exp\left(-\varepsilon/T\right) & T \ll \varepsilon \\ 4T^{-1}/3 & T \gg \varepsilon \end{array}$$

e)

$$\overline{\chi} = \int_0^\infty \chi\left(\varepsilon\right)P\left(\varepsilon\right)d\varepsilon = \mu_B^2 \begin{array}{l} 8T^{-1}\Delta^{-1}\int_T^\infty \exp\left(-\varepsilon/T-\varepsilon/\Delta\right)d\varepsilon \propto \Delta^{-1} \\ \left(4/3\right)T^{-1}\Delta^{-1}\int_0^T \exp\left(-\varepsilon/\Delta\right)d\varepsilon \propto T^{-1}\left[1-\exp\left(-T/\Delta\right)\right] \propto \Delta^{-1} \end{array} \propto \frac{\mu_B^2}{\Delta}$$