Quiz 1: Thermodynamics

1. A gas obeys the equation of state

$$P = \frac{NT}{V} + \frac{N^2 \tau(T)}{V^2}$$

where $\tau(T)$ is a function of the temperature T only. The gas is initially at temperature T and volume V and is expanded isothermally and reversibly to volume $2V$. For this expansion, find

a) The work done;

b) The heat absorbed;

c) The change of internal energy.

Solution

a) The work done in the expansion

$$R = \int_V^{2V} P dV = NT \ln 2 + \frac{N^2 \tau(T)}{2V}$$

b) The heat absorbed in the expansion

$$\begin{aligned} Q = T \triangle S = T \int_V^{2V} \left(\frac{\partial S}{\partial V} \right)_T dV = T \int_V^{2V} \left(\frac{\partial P}{\partial T} \right)_V dV \\ = T \int_V^{2V} \left(\frac{N}{V} + \frac{N^2 \tau'(T)}{V^2} \right) dV = NT \ln 2 + \frac{N^2 T \tau'(T)}{2V} \end{aligned}$$

c) The energy change is

$$\triangle E = Q - R = \frac{N^2}{2V} \left[\tau(T) - T\tau'(T) \right]$$

2. The efficiency of a heat engine is defined as the ratio of the work done by the engine in a single cycle to the received heat, $\eta = |R|/Q$. Evaluate the efficiency η of the Joule cycle consisting of two adiabats and two isobars, at pressures P_1 and $P_2 > P_1$ respectively. You may assume that the heat capacities C_v and C_p are constant and that you know the ratio $\gamma = C_p/C_v$. *Hint*: derive the dependence of E and W on T for constant C_v and C_p.

Solution

For constant heat capacities

$$E = C_v T + E_0,\ W = C_p T + E_0$$
$$PV = W - E = (C_p - C_v)\, T$$

The work done in a cycle $1 \to 2 \to 3 \to 4 \to 1$ (1 and 4 are at P_1, 2 and 3 are at P_2)

$$\begin{aligned} |R| &= C_v \left[(T_3 - T_4) - (T_2 - T_1)\right] + (C_p - C_v)(T_3 - T_2) - (C_p - C_v)(T_4 - T_1) \\ &= C_p \left[(T_3 - T_2) - (T_4 - T_1)\right] \end{aligned}$$

The heat received

$$Q = C_p (T_3 - T_2)$$

In an adiabatic process

$$C_v dT = -PdV = -(C_p - C_v)\, T \frac{dV}{V} \Rightarrow \frac{dT}{T} = (1-\gamma)\frac{dV}{V}$$
$$T \propto V^{(1-\gamma)} \propto T^{(1-\gamma)}/P^{(1-\gamma)} \Rightarrow T \propto P^{(\gamma-1)/\gamma}$$

whereof

$$\frac{T_1}{T_2} = \frac{T_4}{T_3} = \left(\frac{P_1}{P_2}\right)^{(\gamma-1)/\gamma}$$

The efficiency is

$$\eta = \frac{|R|}{Q} = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - \left(\frac{P_1}{P_2}\right)^{(\gamma-1)/\gamma}$$