Assignment 4: Quantum Statistics

1. Calculate the Fermi energy and the total energy of a $1D$, spin-1/2 Fermi gas at $T = 0$. What is the relationship between the two? Repeat your calculation for a $2D$ gas.

2. Calculate the sound velocity

$$u^2 = \left(\frac{\partial P}{\partial \rho}\right)_{T=0}$$

of a spin-1/2 Fermi gas. Also, calculate $(\partial P/\partial \rho)_{T=0}$ for a Bose gas below the Bose-Einstein temperature.

3. For a Bose gas with fixed number of particles and at a given temperature, calculate the critical volume at which the Bose-Einstein condensation takes place. Repeat your analysis in $2D$.

4. For a dispersion relation $\varepsilon \propto |p|^\sigma$, what is the constraint on dimensionality of space D for Bose-Einstein condensation to take place?

5. For a D-dimensional blackbody cavity, what is the functional dependence of energy density on temperature?

6. Consider a harmonic oscillator with equally spaced energy states, $\varepsilon_n = \hbar\omega n$. Find its energy E and entropy at temperature T. Show that the latter can be expressed as

$$S = \langle n+1\rangle \ln\langle n+1\rangle - \langle n\rangle \ln\langle n\rangle$$

where $\langle n\rangle$ is the mean occupancy of a mode $\hbar\omega$, $E = \hbar\omega\langle n\rangle$.

7. In the limits of low and high temperatures, find the functional dependence on T of specific heat of *fermionic* phonons in Debye's theory.

8. Treating neutrino star as a degenerate gas of relativistic fermions, find the relationship between its radius and mass from the condition of equilibrium.

9. Consider a vacuum diode with planar electrodes and negative plate voltage. Derive the formula for the current in the diode, given that the work function is ϕ and the retarding potential is V. Assume that $e(\phi + V) \gg T$. *Hint*: $J = e\langle n v_x\rangle$ and the minimal kinetic energy required to traverse the diode is found from $mu^2/2 = e(\phi + V) + E_F$.