

# Generalized functions

Example: density of a point mass at origin

$$f_\epsilon(x) = \begin{cases} \frac{3}{4\pi\epsilon^3}, & |x| < \epsilon \\ 0, & |x| > \epsilon \end{cases} \xrightarrow{\epsilon \rightarrow 0} \begin{cases} +\infty, & x=0 \\ 0, & x \neq 0 \end{cases}$$

Must have  $\int_{\bar{V}} \delta(x) dx = \begin{cases} 1, & 0 \in \bar{V} \\ 0, & 0 \notin \bar{V} \end{cases}$

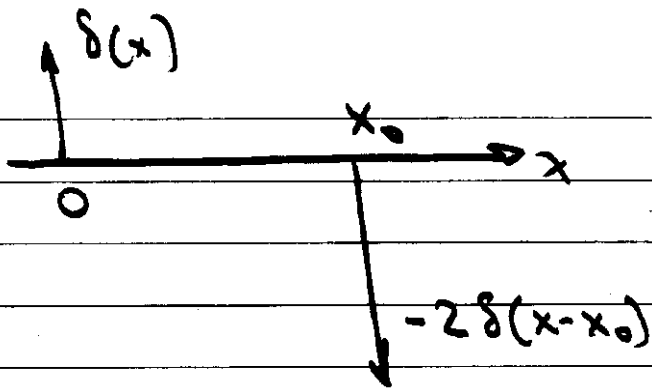
"weak limit"  $f_\epsilon(x) \xrightarrow{\epsilon \rightarrow 0} \delta(x)$

$$\lim_{\epsilon \rightarrow 0} \int f_\epsilon(x) \phi(x) dx = \phi(0) \equiv (\delta, \phi)$$

## Exercise (1D)

Show that

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\sqrt{\pi}\epsilon} e^{-\frac{x^2}{4\epsilon^2}} \\ \frac{1}{\pi x} \sin \frac{x}{\epsilon} \\ \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \\ \frac{\epsilon}{\pi x^2} \sin^2 \frac{x}{\epsilon} \end{cases} \xrightarrow{\epsilon \rightarrow 0} \delta(x)$$



②

Example

Simple (single) layer  $\Gamma$  on surface  $S$   
density  $\rho$

$$(\rho \delta_S, \phi) = \int_S \rho(x) \phi(x) dS$$

Example

Show that

$$\frac{1}{x+i\epsilon} \xrightarrow{\epsilon \rightarrow 0} -i\pi \delta(x) + \mathcal{P} \frac{1}{x}$$

that is

$$\int \frac{q(x)}{x+i\epsilon} dx \xrightarrow{\epsilon \rightarrow 0} -i\pi q(0) + \mathcal{P} \int \frac{q(x)}{x} dx$$

$$\lim_{\epsilon \rightarrow 0} \int \frac{\phi(x)}{x+i\epsilon} dx = \phi(0) \lim_{\epsilon \rightarrow 0} \int \frac{x-i\epsilon}{x^2+\epsilon^2} dx$$

$$+ \lim_{\epsilon \rightarrow 0} \int \frac{x-i\epsilon}{x^2+\epsilon^2} [\phi(x) - \phi(0)] dx$$

odd fn

$$\int \frac{x}{x^2+\epsilon^2} dx = 0$$

$$\int \frac{-i\epsilon}{x^2+\epsilon^2} dx = -i\pi$$

$$\lim_{\epsilon \rightarrow 0} \int \frac{x-i\epsilon}{x^2+\epsilon^2} [\phi(x) - \phi(0)] dx$$

$$= \int \frac{\phi(x) - \phi(0)}{x} dx = \mathcal{P} \int \frac{\phi(x)}{x} dx$$

(if  $\mathcal{P}$ , then  $\int \frac{\phi(0)}{x} dx = 0$ )

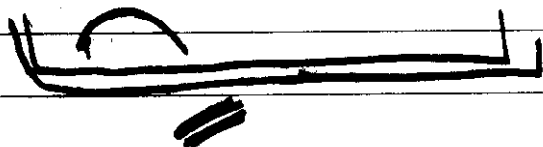
$$\text{So } \boxed{\frac{1}{x \pm i0} = \mp i\pi \delta(x) + \mathcal{P} \frac{1}{x}}$$

# Alternative proof

$$\int \frac{\phi(x)}{x+i\varepsilon} = \int_C \frac{\phi(z)}{z+i\varepsilon}$$



$$= \int_0^x \frac{\phi(x)}{x} + \int_C \frac{\phi(z)}{z} dz$$



$$\lim_{\varepsilon \rightarrow 0} \int_C \frac{\phi(e^{i\phi})}{e^{i\phi}} e^{i\phi} : d\phi = -i\pi \phi(0)$$

## Exercise

Show that

$$\lim_{t \rightarrow \infty} \frac{e^{+ixt}}{x-i0} = \begin{cases} 2\pi i \delta(x) \\ 0 \end{cases}$$

$$\lim_{t \rightarrow \infty} \frac{e^{-ixt}}{x+i0} = \begin{cases} 0 \\ -2\pi i \delta(x) \end{cases}$$

# Derivative

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1, x_2, \dots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

$$D = (D_1, D_2, \dots, D_n), \quad D_j = \frac{\partial}{\partial x_j}$$

other notations

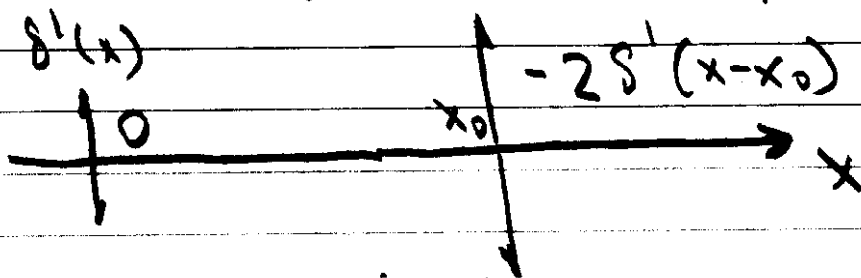
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad \alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

Definition of derivative of generalized fn:

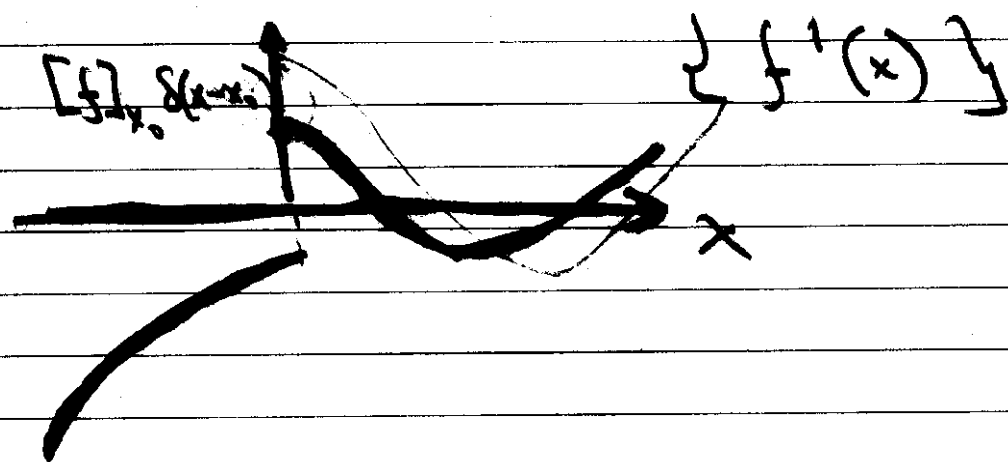
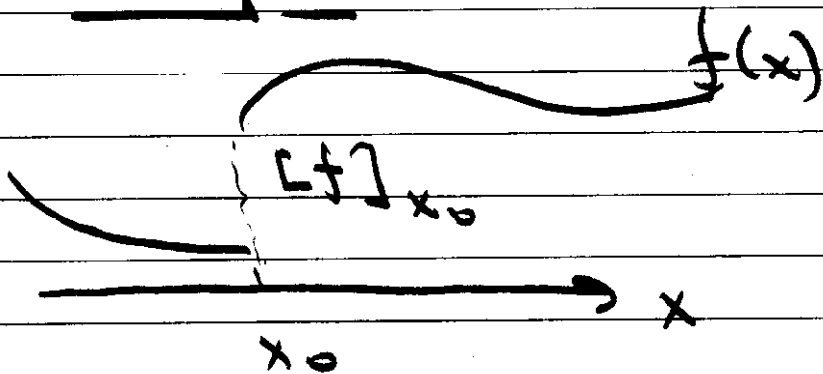
$$(D^\alpha f, \phi) = (-1)^{|\alpha|} (f, D^\alpha \phi)$$

In particular

$$(D^\alpha \delta, \phi) = (-1)^{|\alpha|} D^\alpha \phi(0)$$



## Example



$$[f]_{x_0} = f(x_0+0) - f(x_0-0)$$

$$f'(x) = \{f'(x)\} + [f]_{x_0} \delta(x-x_0)$$

$$(f', \phi) = - (f, \phi') = - \int f(x) \phi'(x) dx$$

$$\rightarrow [f(x_0+0) - f(x_0-0)] \phi(x_0)$$

integrating  
by parts

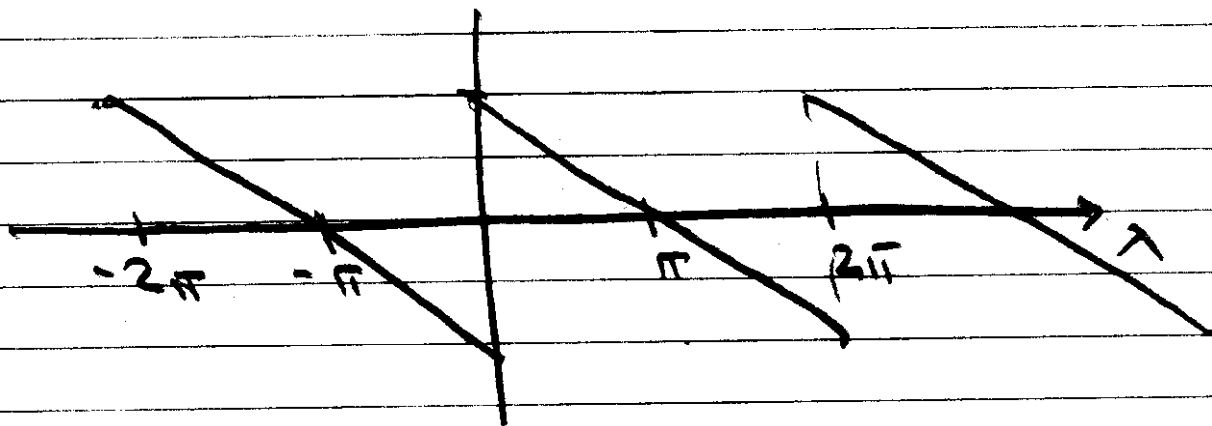
$$+ \int \{f'(x)\} \phi(x) dx$$

In particular, for step-function

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$\theta'(x) = \delta(x)$$

Example



$$f_0(x) = \frac{1}{2} - \frac{x}{2\pi}, \quad x \in [0, 2\pi)$$

$$f_0'(x) = -\frac{1}{2\pi} + \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

Consider  $\int f_0(x) dx = \frac{x}{2} - \frac{x^2}{4\pi}, \quad x \in [0, 2\pi)$

$$\mathcal{F}\left\{f_0\right\} = \frac{1}{2\pi} - \frac{1}{2\pi} \sum_{k \neq 0} \frac{1}{k^2} e^{ikx}$$

$$f_0'(x) = -\frac{1}{2\pi} + \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi) = \frac{1}{2\pi} \sum_{k \neq 0} e^{ikx}$$

$$\left[ \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty} \delta(x-2k\pi) \right]$$


Poisson formula

Example

$$\frac{\partial f}{\partial x_i} = \left\{ \frac{\partial f}{\partial x_i} \right\} + [f]_S \cos(\hat{n} x_i) \delta_S$$

$$\Delta^2 f = \left\{ \Delta^2 f \right\} + \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( [f]_S \cos(\hat{n} x_i) \delta_S \right) + \sum_{i=1}^n \left[ \left\{ \frac{\partial f}{\partial x_i} \right\} \right]_S \cos(\hat{n} x_i) \delta_S$$

$$= \left\{ \Delta^2 f \right\} + \left[ \frac{\partial f}{\partial \hat{n}} \right]_S \delta_S + \frac{\partial}{\partial \hat{n}} ([f]_S \delta_S)$$

  $f=0$  outside

$$= \left\{ \Delta^2 f \right\} - \frac{\partial f}{\partial \hat{n}} \delta_S - \frac{\partial}{\partial \hat{n}} (f \delta_S)$$

applying to  $\phi$

$$\int_G (f \Delta^2 \phi - \phi \Delta^2 f) = \int_S \left( f \frac{\partial \phi}{\partial \hat{n}} - \phi \frac{\partial f}{\partial \hat{n}} \right) \delta_S$$



## Example

Show that  $G(t) = \Theta(t) z(t)$

where  $Lz \equiv z^{(n)} + a_1(t) z^{(n-1)} + \dots + a_n(t) z = 0$

and  $z(0) = z'(0) = \dots = z^{(n-2)}(0) = 0$

$$z^{(n-1)}(0) = 1$$

satisfies  $LG(t) = \delta(t)$

$$G' = \Theta(t) z'(t) + \delta(t) z(t) \quad \propto z(0) = 0$$

$$G^{(n-1)} = \Theta(t) z^{(n-1)}(t)$$

$$G^{(n)} = \delta(t) + \Theta(t) z^{(n)}(t)$$

$$\text{So } LG = \Theta(t) Lz + \delta(t) = \delta(t)$$

For instance

$$G(t) = \Theta(t) e^{-at}, \quad L = \frac{d}{dt} + a$$

$$G(t) = \Theta(t) \frac{\sin at}{a}, \quad L = \frac{d^2}{dt^2} + a^2$$

Retarded GR of diffusion eq. ☺

$$\frac{\partial G}{\partial t} - D \nabla^2 G = \delta(\vec{r}, t) \leftarrow \text{FT}$$

$$\frac{\partial G}{\partial t} + D q^2 G = \delta(t)$$

$$G = \Theta(t) e^{-D q^2 t}$$

$$G(\vec{r}, t) = \Theta(t) \int d\vec{q} e^{i\vec{q} \cdot \vec{r} - D q^2 t}$$

$$= \Theta(t) \prod_{i=1}^d \int dq_i e^{i q_i x_i - D q_i^2 t}$$

$$= \frac{\Theta(t)}{(2\sqrt{\pi D t})^d} e^{-\frac{r^2}{4 D t}}$$

(11)

$$\frac{\partial^2 G}{\partial t^2} - c^2 \nabla^2 G = \delta(\vec{r}, t)$$

$$\delta(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k} \cdot \vec{r}} d\vec{k}$$

$$G(\vec{r}, t) = \frac{1}{(2\pi)^3} \int e^{i\vec{k} \cdot \vec{r}} G(\vec{k}, t) d\vec{k}$$

$$\frac{\partial^2 G}{\partial t^2} + c^2 k^2 G = S(t)$$

$$G(\vec{k}, t) = \frac{\sin ckt}{ck} \Theta(t) \quad (*)$$

Consider 3D,  $S_R$ : sphere of radius  $R$

$$\begin{aligned} \int_{S_R} e^{-i\vec{k} \cdot \vec{r}} dS &= R^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi e^{-ikR \cos\theta} \\ &= 4\pi R \frac{\sin kR}{k} \quad (**)$$

$$G(\vec{k}, t) = \int G(\vec{r}, t) e^{-i\vec{k} \cdot \vec{r}} d\vec{r} \quad (***)$$

From (\*) - (\*\*\*)

$$\boxed{G(\vec{r}, t) = \frac{1}{4\pi c^2 t} \delta_{S_{ct}}(\vec{r})} \quad \left( \begin{array}{l} \text{Notice:} \\ R = ct \end{array} \right)$$

(12)

This can be also written as

$$G_3(\vec{r}, t) = \frac{1}{2\pi c} \delta(c^2 t^2 - r^2)$$

$$= \frac{1}{4\pi r c} \delta(ct - r)$$

$$= \frac{1}{4\pi c^2 t} \delta(ct - r)$$

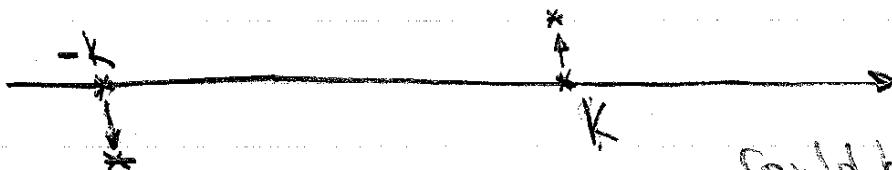
$$(\nabla^2 + k^2) G(\vec{r}) = \delta(\vec{r})$$

$$G(\vec{r}) = \frac{1}{(2\pi)^d} \int G(\vec{q}) e^{i\vec{q} \cdot \vec{r}} d\vec{q}$$

$$\delta(\vec{r}) = \frac{1}{(2\pi)^d} \int e^{i\vec{q} \cdot \vec{r}} d\vec{q}$$

$$G(\vec{q}) = \frac{1}{k^2 - q^2}$$

$$\begin{aligned} G_3(\vec{r}) &= \frac{1}{(2\pi)^3} \int \frac{e^{i\vec{q} \cdot \vec{r}}}{k^2 - q^2} d\vec{q} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty dq q^2 \int_0^\pi d\theta \sin \theta \frac{e^{iqr \cos \theta}}{k^2 - q^2} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty dq \frac{q^2}{k^2 - q^2} \int_{-1}^1 dx e^{iqrx} \\ &= \frac{1}{2\pi^2 r} \int_0^\infty dq \frac{q \sin qr}{k^2 - q^2} = \frac{1}{(2\pi)^2} \int_{-\infty}^\infty dq \frac{q \sin qr}{k^2 - q^2} \\ &= \frac{1}{(2\pi)^2} \frac{1}{2i} \int_{-\infty}^\infty dq \frac{q (e^{iqr} - e^{-iqr})}{k^2 - q^2} \end{aligned}$$

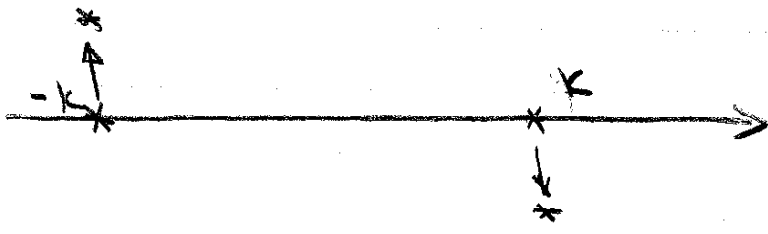


could be other choices

(\*)

$$= -\frac{1}{(2\pi)^2 r} \frac{2\pi i}{2i} \left\{ \text{Res} \left( \frac{e^{iqr}}{p^2 - k^2}, k \right) + \text{Res} \left( \frac{e^{-iqr}}{p^2 - k^2}, -k \right) \right\}$$

$$= -\frac{1}{4\pi r} \left\{ \frac{k e^{ikr}}{2k} + \frac{(-k) e^{ikr}}{(-2k)} \right\} = -\frac{1}{4\pi r} e^{ikr}$$



$$-\frac{1}{4\pi r} e^{-ikr}$$

(\*)

