QM 15-030-710-003 Spring ****
Assignment 8: Time Dependence of States

The due date for this assignment is **.**
Reading assignment: Review Chapters II, V, $VIII$ and XV.

1. Find the time dependence of a plane rotator if initially ($t = 0$) it was in a state whose wave function is given by $\Psi(\phi, t = 0) = A\sin^2\phi$.

   Hint: $\sin^2\phi = (1 - \cos 2\phi)/2$, where the functions in parentheses are eigenfunctions of the Hamiltonian for plane rotator.

   Solution

   $$\psi_m = \frac{1}{\sqrt{\pi}}\cos m\phi$$

   are normalized EFs of a plane rotator with EVs

   $$E_m = \frac{\hbar^2 m^2}{2I}$$

   Consequently,

   $$\Psi(x,t) = \frac{A}{2}\left(1 - \exp\left(-\frac{2i\hbar t}{I}\right)\cos 2\phi\right)$$

2. Find the time dependence of a spatial rotator if initially ($t = 0$) it was in a state whose wave function is given by $\Psi(\theta, t = 0) = A\cos^2\theta$.

   Hint: $\cos^2\theta = \left[1 - \left(1 - 3\cos^2\theta\right)\right]/3$, where the functions in brackets are eigenfunctions of the Hamiltonian for spatial rotator.

   Solution

   $$\Psi(x,t) = \frac{A}{3}\left(1 + \exp\left(-\frac{3i\hbar t}{I}\right)\left(3\cos^2\theta - 1\right)\right)$$

3. Find the velocity $\widehat{\mathbf{v}}$ and acceleration $\widehat{\mathbf{w}}$ operators of a neutral particle with a non-zero magnetic moment (e.g. neutron) in the magnetic field.

   Solution

   $$\widehat{H} = \frac{\widehat{\mathbf{p}}^2}{2m} - \mu\overrightarrow{\mathcal{H}}(\mathbf{r},t)\cdot\widehat{\boldsymbol{\sigma}}$$

   Therefore

   $$\widehat{\mathbf{v}} = \dot{\widehat{\mathbf{r}}} = \frac{i}{\hbar}\left[\widehat{H}, \widehat{\mathbf{r}}\right] = \frac{\widehat{\mathbf{p}}}{m}$$

   and

   $$\widehat{\mathbf{w}} = \dot{\widehat{\mathbf{v}}} = \frac{i}{\hbar}\left[\widehat{H}, \widehat{\mathbf{v}}\right] = -\frac{\nabla\widehat{U}}{m}$$

   where

   $$\widehat{U} = -\widehat{\boldsymbol{\mu}}\cdot\overrightarrow{\mathcal{H}}(\mathbf{r},t) = -\mu\overrightarrow{\mathcal{H}}(\mathbf{r},t)\cdot\widehat{\boldsymbol{\sigma}}$$

   For a particle with arbitrary spin (not necessarily 1/2), $\widehat{\boldsymbol{\sigma}} \to \widehat{\mathbf{s}}/s$.

4. Find the time dependence of the spin function and the mean values of spin projections of a neutral $s = 1/2$ particle with magnetic moment μ in a spatially uniform magnetic field $\overrightarrow{\mathcal{H}}(t) = \mathcal{H}(t)\,\mathbf{n}_0$. Assume that the initial spin function (and spin projections) are known.

   Solution

   $$\Psi(t) = \begin{pmatrix} C_1(0)\exp(i\xi(t)) \\ C_2(0)\exp(-i\xi(t)) \end{pmatrix}$$

where

$$\xi(t) = \frac{\mu}{\hbar}\int_0^t \mathcal{H}(t)\,dt$$

From

$$\overline{\widehat{\mathbf{s}}(t)} = \frac{1}{2}\Psi^*(t)\,\widehat{\boldsymbol{\sigma}}\Psi(t)$$

find

$$\begin{aligned}
\overline{s_x(t)} &= \overline{s_x(0)}\cos 2\xi(t) + \overline{s_y(0)}\sin 2\xi(t) \\
\overline{s_y(t)} &= \overline{s_y(0)}\cos 2\xi(t) - \overline{s_x(0)}\sin 2\xi(t) \\
\overline{s_z(t)} &= s_z(0)
\end{aligned}$$

5. A spin $s = 1/2$ particle with magnetic moment μ is subject to a spatially uniform magnetic field $\overrightarrow{\mathcal{H}}(t)$ such that

$$\begin{aligned}
\mathcal{H}_x(t) &= \mathcal{H}_0\cos\omega_0 t \\
\mathcal{H}_y(t) &= \mathcal{H}_0\sin\omega_0 t \\
\mathcal{H}_z(t) &= \mathcal{H}_1
\end{aligned}$$

Initially ($t = 0$), the particle was in a state with $s_z = 1/2$. Find the probabilities of possible values of s_z at time t. Consider, in particular, the case $|\mathcal{H}_1/\mathcal{H}_0| \ll 1$ and show that "spin flip" for the latter circumstance is of resonance character in terms of dependence on the frequency ω_0. Please, provide a detailed calculation.

Solution

$$\Psi(t) = \frac{1}{2\omega}\begin{pmatrix} [(\omega+\gamma_1)\exp(i\omega t) + (\omega-\gamma_1)\exp(-i\omega t)]\exp(-i\omega_0 t/2) \\ 2i\gamma_2\sin\omega t\exp(-i\omega_0 t/2)\end{pmatrix}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{\mu\mathcal{H}_1}{\hbar} + \frac{\omega_0}{2} \\
\gamma_2 &= \frac{\mu\mathcal{H}_0}{\hbar} \\
\omega &= \sqrt{\gamma_1^2+\gamma_2^2}
\end{aligned}$$

The probability of a pin-flip, that is $s_z = -1/2$, at time t is

$$\begin{aligned}
W(s_z = -1/2,\, t) &= \left(\frac{\gamma_2}{\omega}\right)^2 \sin^2\omega t \\
&= \frac{\mathcal{H}_0^2}{\mathcal{H}_0^2 + (\mathcal{H}_1 + \hbar\omega_0/2\mu)^2}\sin^2\omega t
\end{aligned}$$

This probability is small unless

$$\omega_0 \approx (\omega_0)_{res} \equiv -2\mu\mathcal{H}_1/\hbar\omega_0$$