QM 15-030-710-002 Spring ****
Assignment 7: Motion in Magnetic Field

The due date for this assignment is **.**
Reading assignment: Chapter XV.

1. Show that in the Coulomb gauge the Hamiltonian of a charged particle in the magnetic field

$$\widehat{H} = \frac{1}{2m}\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\left(\mathbf{r}\right)\right)^2$$

can be written as

$$\widehat{H} = \frac{\widehat{\mathbf{p}}^2}{2m} - \frac{e}{cm}\mathbf{A}\cdot\widehat{\mathbf{p}} + \frac{e^2}{2mc^2}\mathbf{A}^2$$

and check that this operator is Hermitian.

Solution

See pp. 456-457 in LL.

2. Find the velocity operator $\widehat{\mathbf{v}}$ of a charged particle in the magnetic field and establish the commutational relationships $[\widehat{v}_i, \widehat{v}_k]$ and $[\widehat{v}_i, \widehat{x}_k]$.

Solution

$$\widehat{\mathbf{v}} = \frac{i}{\hbar}\left[\widehat{H}, \mathbf{r}\right] = \frac{1}{m}\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)$$

$$\begin{aligned} [\widehat{v}_i, \widehat{v}_k] &= \frac{e}{m^2 c}\left([\widehat{p}_k A_i] - [\widehat{p}_i A_k]\right) = \frac{ie\hbar}{m^2 c}\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right) \\ &= \frac{ie\hbar}{m^2 c}\epsilon_{ikl}H_l,\ \mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A} \end{aligned}$$

$$[\widehat{v}_i, \widehat{x}_k] = -\frac{i\hbar}{m}\delta_{ik}$$

3. For a charged particle in a constant homogeneous magnetic field, find the operator of the center of the orbit $\widehat{\rho}_0$ of the transverse (perpendicular to the magnetic field) motion and $\widehat{\rho}_0^2$. Also find the operator $\widehat{\rho}_L^2$ of the squared radius of the orbit. Establish the commutational relationships of these operators with each other and with the Hamiltonian.

Solution

Classically,

$$\rho_L^2 = \frac{v_\perp^2}{\omega_H^2} = \frac{\left(v_x^2 + v_y^2\right)}{\omega_H^2},\ \omega_H = \frac{|e|\,H}{mc}$$

$$\boldsymbol{\omega}\times(\boldsymbol{\rho} - \boldsymbol{\rho}_0) = \mathbf{v}_\perp,\ \boldsymbol{\omega} = \{0, 0, -\omega_H\}$$

that is

$$x_0 = x - \frac{v_y}{\omega},\ y_0 = y + \frac{v_x}{\omega},\ \rho_0^2 = x_0^2 + y_0^2$$

Quantum mechanically,

$$\begin{aligned} \widehat{x}_0 &= \widehat{x} - \frac{\widehat{v}_y}{\omega} = x - \frac{1}{m\omega}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right) \\ \widehat{y}_0 &= \widehat{y} + \frac{\widehat{v}_x}{\omega} \\ \widehat{\rho}_0^2 &= \widehat{x}_0^2 + \widehat{y}_0^2 \\ \widehat{\rho}_L^2 &= \frac{\left(\widehat{v}_x^2 + \widehat{v}_y^2\right)}{\omega_H^2} \end{aligned}$$

Commutational relationships

$$\left[\widehat{H}, \widehat{x}_0\right] = \left[\widehat{H}, \widehat{y}_0\right] = \left[\widehat{H}, \widehat{\rho}_0^2\right] = \left[\widehat{H}, \widehat{\rho}_L^2\right] = 0$$

where, for instance,

$$\begin{aligned}\left[\widehat{H},\widehat{x}_0\right] &= \left[\widehat{H},\widehat{x}\right]-\frac{1}{\omega}\left[\widehat{H},\widehat{v}_y\right]=-i\hbar\widehat{v}_x+\frac{m^2c}{2eH}\left[\widehat{v}_x^2,\widehat{v}_y\right]=-i\hbar\widehat{v}_x+\widehat{v}_x\frac{m^2c}{eH}\left[\widehat{v}_x,\widehat{v}_y\right]=0\\ \widehat{H} &= \frac{m\widehat{\mathbf{v}}^2}{2}\end{aligned}$$

Also

$$\left[\widehat{x}_0,\widehat{y}_0\right]=-\frac{i\hbar c}{eH},\ \left[\widehat{\rho}_0^2,\widehat{\rho}_L^2\right]=0$$

4. Find the eigenvalue spectrum of $\widehat{\rho}_0^2$ and $\widehat{\rho}_L^2$ (see preceding problem).

   Solution

$$\begin{aligned}\widehat{\rho}_L^2 &= \frac{\left(\widehat{v}_x^2+\widehat{v}_y^2\right)}{\omega_H^2}=\frac{2}{\omega_H^2}\frac{|e|\,\hbar H}{m^2c}\frac{1}{2}\left(\widehat{P}^2+\widehat{Q}^2\right)=\frac{2\hbar c}{|e|\,H}\frac{1}{2}\left(\widehat{P}^2+\widehat{Q}^2\right)\\ \widehat{P} &= \sqrt{\frac{m^2c}{|e|\,\hbar H}}\widehat{v}_y,\ \widehat{Q}=\sqrt{\frac{m^2c}{|e|\,\hbar H}}\widehat{v}_x,\ \left[\widehat{P},\widehat{Q}\right]=-i\hbar\end{aligned}$$

   Therefore, eigenvalues of $\widehat{\rho}_L^2$ are

$$\frac{2\hbar c}{|e|\,H}\left(n+\frac{1}{2}\right)$$

   By analogy, and using the result of the previous problem for $[\widehat{x}_0,\widehat{y}_0]$, the spectrum $\widehat{\rho}_0^2$ is the same as $\widehat{\rho}_L^2$.

5. Find the eigenvalues and eigenfunctions of the stationary states of a charged plane rotator (a charged particle moving in a plane at a fixed distance a form the center) in a constant homogeneous magnetic field perpendicular to the plane.

   Solution

   Using radial gauge, the Hamiltonian of the transverse motion is

$$\widehat{H}_t=-\frac{\hbar^2}{2m_e}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{\hbar^2}{2m_e\rho^2}l_z^2-\frac{e\hbar H}{2m_ec}l_z+\frac{e^2H^2\rho^2}{8m_ec^2}$$

   Setting $\rho=a=const$ and denoting $I=ma^2$

$$\widehat{H}_t=-\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}+\frac{ie\hbar H}{2m_ec}\frac{d}{d\phi}+\frac{e^2H^2I}{8m_e^2c^2}$$

   whose eigenfunctions and eigenvalues are, respectively,

$$\begin{aligned}\Psi_m(\phi) &= \frac{1}{\sqrt{2\pi}}\exp(im\phi),\ m\text{: integer}\\ E_m &= \frac{\hbar^2m^2}{2I}-\frac{e\hbar Hm}{2m_ec}+\frac{e^2H^2I}{8m_e^2c^2}\end{aligned}$$

6. Find the eigenvalues and eigenfunctions of the stationary states of a neutral $s=1/2$ particle in a constant homogeneous magnetic field.

   Solution

$$\begin{aligned}\Psi_{ps_z} &= \frac{1}{(2\pi\hbar)^{3/2}}\exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right)\chi_{s_z}\\ \chi_{s_z=1/2} &= \begin{pmatrix}1\\0\end{pmatrix},\ \chi_{s_z=-1/2}=\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}$$

7. Establish the relationship between the mean values of the orbital angular momentum $\widehat{\mathbf{l}}$ and the magnetic moment $\widehat{\boldsymbol{\mu}}$ of a spin-less charged particle in a magnetic field. Show that this relationship is consistent with gauge invariance.

   Solution

   Classically,

$$\boldsymbol{\mu} = \frac{e}{2c}\mathbf{r} \times \mathbf{v} = \frac{e}{2mc}\mathbf{r}\times \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)$$

   Quantum mechanically,

$$\begin{aligned}\widehat{\boldsymbol{\mu}} &= \frac{e}{2c}\mathbf{r}\times\widehat{\mathbf{v}}=\frac{e}{2mc}\mathbf{r}\times\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right) \\ &= \frac{e\hbar}{2mc}\widehat{\mathbf{l}} - \frac{e^2}{2mc^2}\,\mathbf{r} \times \mathbf{A}\end{aligned}$$

   Taking expectation value and using that

$$\Psi^{'} = \Psi \exp\left(\frac{ief}{\hbar c}\right)$$

   when $\mathbf{A}^{'} = \mathbf{A} + \nabla f$, we find that neither $\overline{\widehat{\mathbf{l}}}$ nor $\overline{\mathbf{r} \times \mathbf{A}}$ are gauge invariant, $\overline{\widehat{\boldsymbol{\mu}}}$ is.