QM 15-030-710-002 Winter **
Assignment 6: Spin**

The due date for this assignment is **.**
Reading assignment: Chapter $VIII$.

1. Solving an eigenvalue/eigenfunction problem for a $s = 1/2$ particle, find the spin functions ψ_{s_i} $(i = 1, 2, 3)$ of the states with definite projections of the spin on x, y and z axes.

   Solution

$$\widehat{s}_x \psi_{s_x} = s_x \psi_{s_x}$$

for $\psi_{s_x} = \begin{pmatrix} a \\ b \end{pmatrix}$,

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} b \\ a \end{pmatrix} = s_x \begin{pmatrix} a \\ b \end{pmatrix}$$

so that

$$\begin{aligned} b &= 2as_x \\ a &= 2bs_x \end{aligned}$$

whereof

$$s_x^2 = \frac{1}{4}$$

leading to

$$\begin{aligned} a &= -b,\ s_x = -\frac{1}{2} \\ a &= b,\ s_x = \frac{1}{2} \end{aligned}$$

and, using normalization, $\left\|\psi_{s_x}\right\|^2 = a^2 + b^2 = 1$, gives

$$\psi_{s_x=-\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \psi_{s_x=\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and, analogously,

$$\psi_{s_y=-\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \psi_{s_y=\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\psi_{s_z=-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \psi_{s_z=\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. Find the operator $\widehat{s}_n$ of the spin projection on an arbitrary direction defined by the unit vector $\mathbf{n}$. What is the mean value of the projection on $\mathbf{n}$ in a state with a definite $s_z = 1/2$ (or $-1/2$)? What is the probability to have spin projections $s_n = 1/2$ and $-1/2$ in such states?

   Solution

$$\widehat{s}_n = \mathbf{n} \cdot \mathbf{s} = \frac{1}{2}\mathbf{n} \cdot \boldsymbol{\sigma}$$

using

$$\begin{aligned} \mathbf{n} &= \{\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\} \\ \boldsymbol{\sigma} &= \{\sigma_x,\ \sigma_y,\ \sigma_z\} \end{aligned}$$

one finds

$$\widehat{s}_n = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

and

$$\overline{s}_n = \frac{1}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\cos\theta,\ s_z = \frac{1}{2}$$
$$\overline{s}_n = \frac{1}{2}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\cos\theta,\ s_z = -\frac{1}{2}$$

obviously

$$\begin{aligned} \overline{s}_n &= s_z\cos\theta \\ &= \frac{1}{2}w(+) + \left(-\frac{1}{2}\right)w(-); \end{aligned}$$

while

$$1 = w(+) + w(-)$$

whereof

$$\begin{aligned} w(+) &= \frac{1+2s_z\cos\theta}{2} \\ w(-) &= \frac{1-2s_z\cos\theta}{2} \end{aligned}$$

3. Find the eigenvalues of the operator $\widehat{f} = a + \mathbf{b}\cdot\widehat{\boldsymbol{\sigma}}$.
   Solution

$$f_{1,2} = a \pm b$$

4. Simplify the expression $(\mathbf{a}\cdot\widehat{\boldsymbol{\sigma}})^n$. *Hint*: $\widehat{\sigma}_i\widehat{\sigma}_k = \delta_{ik} + i\varepsilon_{ikl}\widehat{\sigma}_l$.
   Solution

$$(\mathbf{a}\cdot\widehat{\boldsymbol{\sigma}})^2 = a_i\widehat{\sigma}_i a_k\widehat{\sigma}_k = a_i a_i = \mathbf{a}^2 = a^2$$

so that

$$(\mathbf{a}\cdot\widehat{\boldsymbol{\sigma}})^n = \begin{matrix} a^n,\ n \text{ even} \\ a^{n-1}(\mathbf{a}\cdot\widehat{\boldsymbol{\sigma}}),\ n \text{ odd} \end{matrix}$$

5. Find the projection operators $P_{s_z=\pm 1/2}$ to the states with definite projection $s_z = \pm 1/2$ on axis z.
   Notice: such operators are Hermitian and satisfy the relationship $P^2_{s_z=\pm1/2} = P_{s_z=\pm1/2}$.
   Solution

$$P_{s_z=1/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

hence

$$P_{s_z=1/2} = a(1+\sigma_z)$$

also

$$P_{s_z=1/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which gives $a = 1/2$ and

$$P_{s_z=1/2} = \frac{1}{2}(1+\sigma_z)$$

$$\begin{aligned} P^2_{s_z=1/2} &= \frac{1}{4}\left(1+2\sigma_z+\sigma_z^2\right) = \frac{1}{4}(2+2\sigma_z) \\ &= P_{s_z=1/2} \end{aligned}$$

6. For a $s=1/2$ particle, find the transformation law of the spin function $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ under rotation of the coordinate system by an angle φ around the axis whose direction is defined by a unit vector $\mathbf{n}$. Show that the quantity $\phi^*\psi = \phi_1^*\psi_1 + \phi_2^*\psi_2$ remains unchanged under such transformation, that is, it is a scalar.

   Solution

$$\psi^{'} = \begin{pmatrix} \psi_1^{'} \\ \psi_2^{'} \end{pmatrix} = \exp\left(\frac{1}{2} i\boldsymbol{\varphi}\cdot\widehat{\boldsymbol{\sigma}}\right)\psi = \left(\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}\right)v$$

$$\phi^{*\prime} = \begin{pmatrix} \phi_1^{*\prime} & \phi_2^{*\prime} \end{pmatrix} = \phi^*\left(\cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}\widehat{\boldsymbol{\sigma}}\cdot\mathbf{n}\right)$$

   and, consequently,

$$\phi^{*\prime}\psi^{'} = \phi^*\psi$$

7. Show that, in a state of two particles with a definite value of the total spin, the operator $\widehat{\boldsymbol{\sigma}}_1\cdot\widehat{\boldsymbol{\sigma}}_2$ also takes a definite value.

   Solution

$$\widehat{\mathbf{S}}^2 = \frac{1}{4}\left(\widehat{\boldsymbol{\sigma}}_1 + \widehat{\boldsymbol{\sigma}}_2\right)^2$$

   and, using $\widehat{\boldsymbol{\sigma}}^2 = 3$,

$$\widehat{\boldsymbol{\sigma}}_1\cdot\widehat{\boldsymbol{\sigma}}_2 = -3 + 2\widehat{\mathbf{S}}^2$$

   so that the eigenvalues are

$$\lambda_{1,2} = -3 + 2S\left(S+1\right) = \begin{matrix} -3,\ S=0 \\ 1,\ S=1 \end{matrix}$$

8. Using the result of the preceding problem, find the projection operators $P_{S=0,1}$ to the states of a two-particle system, each with spin $s=1/2$, with definite values of the total spin.

   Solution

$$\begin{aligned} P_{S=0} &= \frac{1-\widehat{\boldsymbol{\sigma}}_1\cdot\widehat{\boldsymbol{\sigma}}_2}{4} \\ P_{S=1} &= \frac{3+\widehat{\boldsymbol{\sigma}}_1\cdot\widehat{\boldsymbol{\sigma}}_2}{4} \end{aligned}$$

9. Find eigenfunctions and eigenvalues of the operator $a\left(\widehat{\sigma}_{1z} + \widehat{\sigma}_{2z}\right) + b\widehat{\boldsymbol{\sigma}}_1\cdot\widehat{\boldsymbol{\sigma}}_2$.

   Solution

$$\begin{aligned} a\left(\widehat{\sigma}_{1z} + \widehat{\sigma}_{2z}\right) + b\widehat{\boldsymbol{\sigma}}_1\cdot\widehat{\boldsymbol{\sigma}}_2 &= 2a\widehat{S}_z + b\left(-3 + 2\widehat{\mathbf{S}}^2\right) \\ &\rightarrow 2aS_z - 3b + 2bS\left(S+1\right) \end{aligned}$$

10. In a state of a $s=1/2$ particle with definite values l, m, s_z, find the probabilities of different values of $\mathbf{j} = \mathbf{l} + \mathbf{s}$.

    Solution

$$\begin{aligned} \overline{\widehat{\mathbf{j}}^2} &= l\left(l+1\right) + s\left(s+1\right) + 2ms_z = l\left(l+1\right) + \frac{3}{4} + 2ms_z \\ &= w\left(+\right)\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right) + w\left(-\right)\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right) \end{aligned}$$

    while

$$w\left(+\right) + w\left(-\right) = 1$$

where

$$
\begin{aligned}
w(+) &\equiv w\left(j = l + \frac{1}{2}\right) \\
w(-) &\equiv w\left(j = l - \frac{1}{2}\right)
\end{aligned}
$$

whereof

$$
\begin{aligned}
w(+) &= \frac{l + 2ms_z + 1}{2l + 1} \\
w(-) &= \frac{l - 2ms_z}{2l + 1}
\end{aligned}
$$

11. Helicity is defined as a projection of a spin to the direction of the momentum. For a $s = 1/2$ particle, find the wavefunctions $\psi_{\mathbf{p},\lambda}$ of the states with definite momentum $\mathbf{p}$ and helicity $\lambda = \pm 1/2$.

Solution

$$
\begin{aligned}
\psi_{\mathbf{p},\lambda=1/2} &= \frac{\exp\left(i\mathbf{p}\cdot\mathbf{r}/\hbar\right)}{(2\pi\hbar)^{3/2}} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix} \\
\psi_{\mathbf{p},\lambda=-1/2} &= \frac{\exp\left(i\mathbf{p}\cdot\mathbf{r}/\hbar\right)}{(2\pi\hbar)^{3/2}} \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)\exp(i\phi) \end{pmatrix}
\end{aligned}
$$

where θ and ϕ are the azimuthal and polar angle defining the direction of $\mathbf{p}$.

12. Find the form of the helicity operator and show that it commutes with $\widehat{\mathbf{j}} = \widehat{\mathbf{l}} + \widehat{\mathbf{s}}$.

Solution

$$
\widehat{\lambda} = \widehat{\mathbf{s}} \cdot \frac{\mathbf{p}}{p}
$$

and can be easily shown to commute with $\widehat{\mathbf{j}}$. However, this is already obvious on the grounds that $\widehat{\lambda}$ is a scalar operator. Notice that $\widehat{\lambda}$ can also be written as

$$
\widehat{\lambda} = \widehat{\mathbf{j}} \cdot \frac{\mathbf{p}}{p}
$$

since $\widehat{\mathbf{l}} \cdot \mathbf{p} = 0$.