QM 15-030-710-002 Winter ****
Assignment 4: Motion in Central Field

The due date for this assignment is **.**
Reading assignment: Chapter V.

1. Find the energy levels and the normalized wave functions of the stationary states of a plane harmonic oscillator $U(\rho) = k\rho^2/2$ and determine the degeneracy of such states.

   Hint: Use separation of variables in Cartesian coordinates and label the states $\psi_{n_1 n_2}$, where n_1 and n_2 are the quantum numbers of 1D oscillators.

   Solution

$$\begin{aligned} \psi_{n_1 n_2}(x,y) &= \psi_{n_1}(x)\,\psi_{n_2}(y) \\ E_N &= \hbar\omega\,(N+1) \end{aligned}$$

   where $N = n_1 + n_2 = 0, 1, 2, \ldots$ and $n_1, n_2 = 0, 1, 2, \ldots$. There are $N+1$ linearly independent WFs $\psi_{n_1 n_2}(x,y)$, where $n_1 = 0, 1, 2, \ldots, N$ while $n_2 = N, N-1, N-2, \ldots, 0$, corresponding to the energy level E_N.

2. In a stationary state ψ_{11} (see preceding problem) of a plane harmonic oscillator, find the probabilities of various possible projections of the angular momentum along the axis perpendicular to the plane of oscillations.

   Hint: Express the WF in polar coordinates and use $\sin(2\phi) = \left(e^{2i\phi} - e^{-2i\phi}\right)/2i$.

   Solution

$$\begin{aligned} \psi_{11}(x,y) &= \frac{2xy}{\sqrt{\pi}a^3}\exp\left(-\frac{x^2+y^2}{2a^2}\right) = \frac{2\cos\phi\sin\phi}{\sqrt{\pi}a^3}\rho^2\exp\left(-\frac{\rho^2}{2a^2}\right) \\ &= \frac{\left(e^{2i\phi} - e^{-2i\phi}\right)}{2i\sqrt{\pi}a^3}\rho^2\exp\left(-\frac{\rho^2}{2a^2}\right) \end{aligned}$$

   That is $m = 2$ or $m = -2$ with equal probability 1/2.

3. Find the energies of the discrete spectrum levels in a 2D field $U(\rho) = -\alpha/\rho$ and their degeneracy. Compare with the case of the Coulomb field $U(r) = -\alpha/r$.

   Solution

   SE for the radial part of the WF $\psi_{n_\rho m} = \chi_{n_\rho |m|}\exp(im\phi)$,

$$-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right]\chi - \frac{\alpha}{\rho}\chi = E\chi$$

   is transformed via substitution $\chi = u/\sqrt{\rho}$ into

$$\frac{\hbar^2}{2\mu}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2 - 1/4}{\rho^2}\right]u + \left(\frac{\alpha}{\rho} + E\right)u = 0$$

   which is identical to SE in a Coulomb field with the substitution $m^2 - 1/4 \longleftrightarrow l(l+1)$ or $|m| \longleftrightarrow l + 1/2$. Therefore,

$$E_{n_\rho |m|} = -\frac{\mu\alpha^2}{2\hbar^2\left(n_\rho + |m| + 1/2\right)^2} \equiv -\frac{\mu\alpha^2}{2\hbar^2\left(N - 1/2\right)^2}$$

   where the energy depends only on the combination $n_\rho + |m| \equiv N - 1$ (N is the "principal" quantum number in analogy with the Coulomb field). The degeneracy of the energy level E_N is

$$g(N) = 1 + \sum_{|m|=1}^{N} 2 = 2N - 1$$

4. Find an approximate energy of the ground state of a 2D oscillator using the trial function $\psi(\rho) = C\exp(-\alpha\rho)$ and the variational method.

   Solution

   Normalization yields $C = 2\alpha^2/\pi$.

$$\begin{aligned}\overline{T} &= \frac{\hbar^2}{2\mu}\int\left|\frac{d}{d\rho}\psi(\rho)\right|^2 2\pi\rho d\rho = \frac{\hbar^2\alpha^2}{2\mu}\\ \overline{U} &= \overline{\frac{k\rho^2}{2}} = \int\frac{k\rho^2}{2}|\psi(\rho)|^2 2\pi\rho d\rho = \frac{3}{4}\frac{k}{\alpha^2}\end{aligned}$$

   Minimizing $\overline{E} = \overline{T} + \overline{U}$ with respect to α, we find

$$E_0 \approx 1.22\hbar\omega$$

   which should be compared to the exact value $E_0 = \hbar\omega$.

5. Find the energy levels and the normalized wave functions of the stationary states of a spherical harmonic oscillator $U(r) = kr^2/2$ and determine the degeneracy of such states.

   Solution

$$\begin{aligned}\psi_{n_1n_2n_3}(x,y) &= \psi_{n_1}(x)\psi_{n_2}(y)\psi_{n_3}(z)\\ E_N &= \hbar\omega(N+3/2)\end{aligned}$$

   where $N = n_1 + n_2 + n_3 = 0, 1, 2, \ldots$ and $n_1, n_2, n_3 = 0, 1, 2, \ldots$. The degeneracy of the energy level E_N is

$$G(N) = \sum_{n_1=0}^{N}(N - n_1 + 1) = \frac{(N+1)(N+2)}{2}$$

   since for any n_1 the degeneracy is $N - n_1 + 1$, as discussed in the planar case above.

6. Find the effective (mean) potential $\varphi(r)$ acting on a charged particle in a hydrogen atom in the ground state, neglecting the polarization of the latter.

   Hint: Use

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

   where $r_<$ and $r_>$ denote the lesser and the greater, respectively, of r and r', and the orthogonality of Legendre polynomials.

   Solution

$$\begin{aligned}\rho(\mathbf{r}) &= e\delta(\mathbf{r}) - e|\psi_0(\mathbf{r})|^2\\ \varphi(\mathbf{r}) &= \int\frac{\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \frac{e}{r} - e\int\frac{|\psi_0(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\end{aligned}$$

   where $\psi_0(\mathbf{r})$ is the ground state electron WF in a H -atom. Using hint and the fact that only P_0 will contribute due to spherical symmetry of ψ_0 and orthogonality of Legendre polynomials, we obtain

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{e}{r} - \frac{4e}{a^3}\left[\frac{1}{r}\int_0^r\exp(-2r'/a)(r')^2dr' + \int_r^\infty\exp(-2r'/a)r'dr'\right]\\ &= e\left(\frac{1}{r}+\frac{1}{a}\right)\exp(-2r/a)\end{aligned}$$

   In the limiting cases $r \to 0$ and $r \to \infty$ one finds, respectively, the Coulomb field e/r of the proton and the almost completely screened field $(e/a)\exp(-2r/a)$ of the proton by the electron.

7. Find an approximate energy of the ground state of a particle in the Coulomb field $U(r) = -\alpha/r$ using the trial function $\psi(r) = C\exp\left(-\varkappa^2 r^2\right)$ and the variational method. Compare with the exact result.

   Solution

   Normalization yields $C = \left(2\varkappa^2/\pi\right)^{3/2}$.

$$\begin{aligned}\overline{T} &= \frac{\hbar^2}{2\mu}\int \left|\nabla\psi(r)\right|^2 dV = \frac{3\hbar^2\varkappa^2}{2\mu} \\ \overline{U} &= \overline{-\frac{\alpha}{r}} = -\alpha\int \frac{1}{r}\left|\psi(r)\right|^2 dV = -\sqrt{\frac{8}{\pi}}\alpha\varkappa\end{aligned}$$

   Minimizing $\overline{E} = \overline{T} + \overline{U}$ with respect to $\varkappa$, we find

$$E_0 \approx -\frac{3}{4\pi}\frac{\mu\alpha^2}{\hbar^2} \approx -0.42\frac{\mu\alpha^2}{\hbar^2}$$

   which should be compared to the exact value $E_0 = -\mu\alpha^2/2\hbar^2$.

8. Find the Green's function $G_E(\mathbf{r}, \mathbf{r}')$ of the Schrödinger's equation for a free particle with $E < 0$,

$$\left(\widehat{H} - E\right)G_E \equiv -\frac{\hbar^2}{2m}\nabla^2 G_E - EG_E = \delta(\mathbf{r} - \mathbf{r}')$$

   such that it decays when $r \to \infty$. Use the latter to derive an integral form of the Schrödinger's equation for discrete spectrum states of a particle in the field $U(r)$ that decays for $r \to \infty$.

   Solution

   Using notation $E = -\hbar^2\varkappa^2/2\mu$, GF (see e.g. Vladimirov, V. S., Equations of mathematical physics, on reserve),

$$G_E = \frac{\mu}{2\pi\hbar^2}\frac{\exp\left(-\varkappa\left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}$$

   and SE can be re-written as

$$\psi(\mathbf{r}) = -\int G_E(\mathbf{r}, \mathbf{r}')\,U(\mathbf{r}')\,\psi(\mathbf{r}')\,d\mathbf{r}'$$