Assignment # 3 - Solutions

(3.3) Problem 1

$$\overline{\hat{\vec{l}}^{\,2}} = \overline{\hat{l}_x^2} + \overline{\hat{l}_y^2} + \overline{\hat{l}_z^2} = 3\,\overline{l_z^2}$$

$$\overline{l_z^2} = \frac{1}{2l+1}\sum_{m=-l}^{l} m^2 = \frac{2}{2l+1}\sum_{m=0}^{l} m^2 = \frac{l(l+1)}{3}$$

$$\sum_{m=0}^{l} m^2 = \frac{d^2}{d\alpha^2}\sum_{m=0}^{l} e^{\alpha m}\Big|_{\alpha=0} = \frac{d^2}{d\alpha^2}\,\frac{1-e^{\alpha(l+1)}}{1-e^{\alpha}}\Big|_{\alpha}$$

$$= \frac{l(l+1)(2l+1)}{6}$$

(3.11) Problem 2

$$\hat{l}_z\hat{l}_\pm = \hat{l}_\pm(\hat{l}_z \pm 1)$$

$$\hat{l}_z(\hat{l}_\pm \psi_m) = \hat{l}_\pm\left[(\hat{l}_z \pm 1)\psi_m\right] = (m\pm 1)(\hat{l}_\pm \psi_m)$$

(3.12) Problem 3

$$\langle m|\hat{l}_+ m\rangle \propto \langle m|m+1\rangle = 0 \qquad (*)$$

$$\langle m|\hat{l}_+^2|m\rangle = 0 \qquad (**)$$

From $(*)$, $\overline{l_x + i l_y} = 0$, $\overline{l_x} = \overline{l_y} = 0$

From $(**)$, $\overline{\hat{l}_x^2 - \hat{l}_y^2 + i(\hat{l}_y\hat{l}_x + \hat{l}_x\hat{l}_y)} = 0$

$\Rightarrow$ $\overline{l_x^2} = \overline{l_y^2}$ and $\overline{\hat{l}_x\hat{l}_y} = -\overline{\hat{l}_y\hat{l}_x}$

Using $[\hat{l}_x, \hat{l}_y] = i\hat{l}_z$, $\overline{\hat{l}_x\hat{l}_y} - \overline{\hat{l}_y\hat{l}_x} = im$

Consequently, $\overline{\hat{l}_x\hat{l}_y} = -\overline{\hat{l}_y\hat{l}_x} = \frac{im}{2}$

(3.14) Problem 4

$$\hat{l}_{\tilde{z}} = \hat{l}_z\cos\alpha + \hat{l}_x\sin\alpha\cos\beta + \hat{l}_y\sin\alpha\sin\beta$$

Averaging over a state ψ_{lm},

$$\overline{l_{\tilde{z}}} = m\cos\alpha \quad (\text{since } \overline{l_x} = \overline{l_y} = 0)$$

$$\overline{l_{\tilde{z}}^2} = \overline{l_z^2}\cos^2\alpha + \overline{l_x^2}\sin^2\alpha\cos^2\beta + \overline{l_y^2}\sin^2\alpha\sin^2\beta$$

$$= \overline{l_z^2}\cos^2\alpha + \overline{l_x^2}\sin^2\alpha = m^2\cos^2\alpha + \frac{l(l+1)-m^2}{2}\sin^2\alpha$$

Problem 4 (continued)

where we used $\overline{l_x^2} = \overline{l_y^2}$ and $\overline{l_x^2 + l_y^2 + l_z^2} = l(l+1)$

$$\overline{(\Delta l_{\tilde{z}})^2} = \overline{l_{\tilde{z}}^2} - \overline{l_{\tilde{z}}}^2$$

$$= \cancel{m^2\cos^2\alpha} + \frac{l(l+1)-m^2}{2}\sin^2\alpha - \cancel{m^2\cos^2\alpha}$$

In evaluation of $\overline{l_{\tilde{z}}^2}$, we used the fact that $\overline{l_x} = \overline{l_y} = 0$, and $\overline{\hat{l}_x \hat{l}_y + \hat{l}_y \hat{l}_x} = 0$

3.22 Problem 5 [redacted]

Rewrite $Y_{10}(\theta, \varphi)$ as $Y_{10} = i\sqrt{\frac{3}{4\pi}}\cos\theta$

$$= i\sqrt{\frac{3}{4\pi}}\,\frac{z}{r} = i\sqrt{\frac{3}{4\pi}}\,\frac{(\vec{k}\cdot\vec{r})}{r}, \quad \vec{k} = (0,0,1)$$

Since all directions are equivalent,

$$\psi_{\tilde{m}=0} = i\sqrt{\frac{3}{4\pi}}\,\frac{(\vec{n}\cdot\vec{r})}{r}, \text{ where } \vec{n} = \{\sin\alpha\cos\beta, \dots\}$$

$$\vec{r} = \{\sin\theta\cos\varphi, \dots\}$$

$$= i\sqrt{\frac{3}{4\pi}}\left[\sin\theta\sin\alpha\cos(\beta-\varphi) + \cos\theta\cos\alpha\right]$$

(3.28) Problem 6

Since $m = 0, \pm 1$ for $l = 1$,

$$\hat{l}_x^3 = \hat{l}_x, \quad \hat{l}_y^3 = \hat{l}_y$$

which can be also easily by matrix multiplication (in matrix representation)

Since also

$$\overline{l_x} = \overline{l_y} = 0 \quad \text{and} \quad \overline{l_x^2} = \overline{l_y^2} = \frac{\overset{l(l+1)|_{l=1}}{\overset{\downarrow}{2}} - m^2}{2}$$

we find

$$\overline{l_x^n} = \overline{l_y^n} = \begin{cases} 0, & n \text{ odd} \\ \frac{2-m^2}{2}, & n \text{ even} \end{cases}$$

(3.35) Problem 7

$\hat{\vec{V}} = \hat{\vec{l}}_1 \times \hat{\vec{l}}_2$, $\hat{V}_i = \varepsilon_{ikl} \hat{l}_{1k} \hat{l}_{2l}$ – Hermitian

Using $[\hat{l}_i, \hat{l}_k] = i \varepsilon_{ikl} \hat{l}_l$ and $\varepsilon_{ikl}\varepsilon_{\alpha\beta l} = \delta_{i\alpha}\delta_{k\beta} - \delta_{i\beta}\delta_{k\alpha}$

$$\hat{V}^2 = \hat{V}_i \hat{V}_i = \hat{\vec{l}}_1^{\,2} \hat{\vec{l}}_2^{\,2} - (\hat{\vec{l}}_1 \cdot \hat{\vec{l}}_2)^2 - \underbrace{(\hat{\vec{l}}_1 \cdot \hat{\vec{l}}_2)}_{\text{"extra" term, due to}}$$

3.37-38

Problem 8

A. $\max\{|l_1-l_2|, |m_1+m_2|\} \le L \le l_1+l_2$

$$\overline{L_x} = \overline{L_y} = 0, \quad \overline{L_z} = m_1+m_2$$

Since $\overline{l_x} = \overline{l_y} = 0$ and

$$\hat{L}^2 = \hat{\vec{l}}_1^{\,2} + \hat{\vec{l}}_2^{\,2} + 2\hat{\vec{l}}_1 \cdot \hat{\vec{l}}_2$$

we find $\overline{\hat{L}^2} = l_1(l_1+1) + l_2(l_2+1) + 2m_1m_2$

B. For $m_1 = l_1$ and $m_2 = l_2 - 1$,

$L = l_1 + l_2$ or $L = l_1 + l_2 - 1$

If $w(L)$ is the probability of L,

$w(l_1+l_2) + w(l_2+l_1-1) = 1$ and

$$\overline{\hat{L}^2} = \sum_L L(L+1)w(L) = (l_1+l_2)(l_1+l_2+1)w(l_1+l_2) + (l_1+l_2)(l_1+l_2-1)w(l_1+l_2-1)$$

$$\parallel$$

$l_1(l_1+1) + l_2(l_2+1) + 2l_1(l_2-1)$ (from part A)

Combining, $w(l_1+l_2) = \frac{l_2}{l_1+l_2}$, $w(l_1+l_2-1) = \frac{l_1}{l_1+l_2}$