Assignment #2 – Solutions

Problem 1

(2.4)

a) $A^2 \int_0^a x^2 (x-a)^2\, dx = 1$, $A = \sqrt{\frac{30}{a^5}}$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin \frac{\pi (n+1) x}{a}, & 0 < x < a \\ 0, & x < 0, x > a \end{cases}$$

$$C_n = A\sqrt{\frac{2}{a}} \int_0^a x(x-a) \sin \frac{\pi (n+1) x}{a} = -\frac{4\sqrt{15}}{\pi^3} \frac{1+(-1)^n}{(n+1)^3}$$

$$w_n = |C_n|^2 = \frac{240}{\pi^6} \frac{\left[1+(-1)^n\right]^2}{(n+1)^6}$$

$$E_n = \frac{\hbar^2 \pi^2 (n+1)^2}{2ma^2}$$

$$\overline{E} = \sum_{n=0}^{\infty} E_n w_n = E_0 \frac{240}{\pi^6} \sum_{k=0}^{\infty} \frac{4}{(2k+1)^4} = \frac{10}{\pi^2} E_0$$

$$\approx 1{,}014\, E_0$$

$$\overline{E^2} = \sum_{n=0}^{\infty} E_n^2 w_n = E_0^2 \frac{240}{\pi^6} \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2} = E_0^2 \frac{120}{\pi^4}$$

$$\sqrt{\overline{(\Delta E)^2}} = \frac{2\sqrt{5}}{\pi^2} E_0$$

Problem 1 (continued)

b) $B^2 \int_0^a \sin^4\left(\frac{\pi x}{a}\right) dx = 1, \quad B = \sqrt{\frac{8}{3a}}$

$$C_n = B\sqrt{\frac{2}{a}} \int_0^a \sin^2\left(\frac{\pi x}{a}\right) \sin \frac{\pi (n+1)x}{a} dx$$

$$= -\frac{8\left[1+(-1)^n\right]}{\sqrt{3}\pi(n-1)(n+1)(n+3)} = -\frac{8\left[1+(-1)^n\right]}{\sqrt{3}\pi(n+1)\left[(n+1)^2-4\right]}$$

$$w_n = \frac{64}{3\pi^2} \frac{\left[1+(-1)^n\right]}{(n+1)^2\left[(n+1)^2-4\right]^2}$$

$$\overline{E} = E_0 \frac{256}{3\pi^2} \sum_{k=0}^{\infty} \frac{1}{\left[(2k+1)^2-4\right]^2} = \frac{4}{3} E_0$$

$$\overline{E^2} = E_0^2 \frac{256}{3\pi^2} \sum_{k=0}^{\infty} \frac{(2k+1)^2}{\left[(2k+1)^2-4\right]^2} = \frac{16}{3} E_0^2$$

$$\sqrt{\overline{(\Delta E)^2}} = \frac{4\sqrt{2}}{3} E_0$$

2.7 Problem 2

Even states $\psi(x) = \begin{cases} A\cos\left[\sqrt{2m(U_0-|E|)/\hbar^2}\, x\right], & |x|\le a \\ B\exp\left[-\sqrt{}\, x\right], & |x|>a \end{cases}$

From continuity conditions
$$\psi(x\to a+0) = \psi(x\to a-0)$$
$$\psi'(x\to a+0) = \psi'(x\to a-0)$$

find equ. for discrete spectrum
$$\sqrt{U_0-|E|}\,\tan\sqrt{2m(U_0-|E|)a^2/\hbar^2} = \sqrt{|E|}$$

notations: $\xi = \dfrac{2mU_0a^2}{\hbar^2}$, $x = \dfrac{2m|E|a^2}{\hbar^2}$, $0<y=\dfrac{x}{\xi}<1$

$$\sqrt{\frac{1-y}{y}}\,\tan\left(\sqrt{\xi}\sqrt{1-y}\right) = 1 \qquad (+)$$

(+) has N_+ solutions if $(N_+-1)\pi < \sqrt{\xi} < N_+\pi$
since tan becomes ∞ and ϕ N_+ times, e.g.
$$\sqrt{\xi}\sqrt{1-y_\infty} = (N_+-\tfrac{1}{2})\pi,\ (N_+-\tfrac{3}{2})\pi, \dots, \frac{\pi}{2}$$

(+) always has at least one solution

Odd states
$$\sqrt{\frac{1-y}{y}}\,\cot\left(\sqrt{\xi}\sqrt{1-y}\right) = -1 \qquad (-)$$

Problem 2 (continued)

(−) has N_- solutions if $(N_- - \frac{1}{2})\pi < \sqrt{\xi} < (N_- + \frac{1}{2})\pi$
i.e. the states of discrete spectrum exist only when

$$\sqrt{\xi} \geq \frac{\pi}{2} \quad \text{or} \quad U_0 \geq \frac{\pi^2 \hbar^2}{8ma^2}$$

As ξ (U_0) increases, the number of discrete states increases with alterating even-odd, the lowest being even.

(+) and (−) can be combined:

$$(N-1)\frac{\pi}{2} < \sqrt{\xi} \leq N\frac{\pi}{2}, \quad N = N_+ + N_-$$

"=" signals appearance of a new state

For deep well, $\xi \gg 1$, and states near bottom, $\Delta \equiv 1 - y \ll 1$

$$\sqrt{\Delta}\tan\sqrt{\xi\Delta} \simeq 1$$

0'th approx. $\sqrt{\xi\Delta_0} = \frac{\pi}{2}(n+1)$, $n = 0, 1, \ldots$

1'st appox.

$\sqrt{\xi\Delta} \simeq \frac{\pi}{2}(n+1) + \alpha$, $\sqrt{\Delta}\left(-\frac{1}{\alpha}\right) \simeq 1$, $\alpha \simeq -\sqrt{\Delta}$

$$\sqrt{\xi\Delta_1} = \frac{\pi}{2}(n+1) - \sqrt{\Delta_0} = \frac{\pi}{2}(n+1)\left(1 - \sqrt{\frac{1}{\xi}}\right)$$

$$E_n' = U_0 - |E_n| = \frac{\hbar^2\pi^2(n+1)^2}{8ma^2}\left(1 - \sqrt{\frac{1}{\xi}}\right)$$

2.11 Problem 3

$$-\frac{\hbar^2}{2m}\psi'' - \alpha\delta(x)\psi = E\psi = -\frac{\hbar^2\varkappa^2}{2m}, \quad \varkappa > 0$$

$$\psi(x) = A e^{-\varkappa|x|}$$

Integrate SE around ϕ:

$$-\frac{\hbar^2}{2m}\left[\psi'(+0) - \psi'(-0)\right] - \alpha\psi(0) = 0$$

$$\psi' = -\varkappa A |x|' e^{-\varkappa|x|}$$

$$\psi'(+0) = -\varkappa A, \quad \psi'(-0) = \varkappa A$$

$$-\frac{\hbar^2}{2m}(-2\varkappa A) - \alpha A = 0$$

$$\varkappa_0 = \frac{m\alpha}{\hbar^2}, \quad E_0 = -\frac{m\alpha^2}{2\hbar^2}$$

$$A^{-1} = \sqrt{2\int_0^\infty dx\, e^{-2\varkappa_0 x}} = \sqrt{\frac{1}{\varkappa_0}} = \frac{\hbar}{\sqrt{m\alpha}}$$

$$\bar{U} = -\alpha\int_{-\infty}^{\infty}\delta(x)\psi_0^2(x)\,dx = -\frac{m\alpha^2}{\hbar^2} = 2E_0$$

$$\bar{T} = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\, \psi_0''\psi_0 = E_0 - \bar{U} = -E_0$$

2.14

Problem 4

Notations: $E = \varepsilon \dfrac{\hbar^2}{2ma^2}$, $z = \dfrac{x}{a}$, $\xi = \dfrac{2mU_0a^2}{\hbar^2}$

$$-\frac{d^2\psi}{dz^2} + \xi f(z)\psi = \varepsilon \psi \qquad (*)$$

Let $\varepsilon_n(\xi)$ and $\hat{\psi}_n(z,\xi)$ be EV and normalized EF of the dimensionless SE $(*)$

Then

$$\psi_n(x) = \hat{\psi}_n\left(\frac{x}{a}\right)\frac{1}{\sqrt{a}}$$

$$E_n = \varepsilon_n \frac{\hbar^2}{2ma^2} = \varepsilon_n \frac{U_0}{\xi}$$

$$\overline{x_{nn}} = a\,\overline{z}_{nn}, \quad \overline{(\Delta x)^2} = a^2\,\overline{\Delta z^2_{nn}}$$

ε_n, $\overline{z}$, and $\overline{(\Delta z)^2}$ depend only on ξ

For $\xi = \text{const}$, $E_n \propto U_0$

$\overline{x} \propto a$

$\overline{(\Delta x)^2} \propto a^2$

(2.15) Problem 5

Potential energy: $U(z) = \begin{cases} mgz, & z > 0 \\ \infty, & z < 0 \end{cases}$

Ignore (x,y) motion (free particle motion)

$$-\frac{\hbar^2}{2m}\psi''(z) + mgz\,\psi(z) = E\,\psi(z)$$

$$-\frac{\hbar^2}{2m}\psi'' + mg\left(z - \frac{E}{mg}\right)\psi = 0$$

Change of variable: $x = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}\left(z - \frac{E}{mg}\right)$

$$\psi''(x) - x\,\psi(x) = 0 \Rightarrow \psi(x) = Ai(x)$$

$$\psi(z) = \text{const}\; Ai\left[\left(\frac{2m^2g}{\hbar^2}\right)^{1/3}\left(z - \frac{E}{mg}\right)\right]$$

Level quantization from $\psi(z=0) = 0$

$$E_n = \left(\frac{mg^2\hbar^2}{2}\right)\alpha_{n+1}, \quad n = 0, 1, \ldots$$

α_k are roots of $Ai(-\alpha_k) = 0$

Problem 6

(a) $\int_0^\infty \psi^2 dx = 1 = A^2 \int_0^\infty x^2 e^{-2\alpha x} dx$

$$= \frac{A^2}{(2\alpha)^3}\int_0^\infty y^2 e^{-y} dy = \frac{A^2}{(2\alpha)^3}\frac{d^2}{d\beta^2}\int_0^\infty e^{-\beta y} dy\Big|_{\beta=1}$$

$$= \frac{A^2}{(2\alpha)^3}\,2 = \frac{A^2}{4\alpha^3} \Rightarrow A^2 = 4\alpha^3$$

$$\bar{U} = kA^2\int_0^\infty x^3 e^{-2\alpha x} dx = \frac{kA^2}{(2\alpha)^4}\int_0^\infty y^3 e^{-y} dy$$

$$= \frac{6kA^2}{(2\alpha)^4} = \frac{3k}{2\alpha}$$

$$\bar{T} = \frac{\hbar^2}{2m}\int_0^\infty \left(\frac{d\psi}{dx}\right)^2 dx = \frac{\hbar^2 A^2}{2m}\int_0^\infty \left(1-\alpha x\right)^2 e^{-2\alpha x} dx$$

$$= \frac{\hbar^2 A^2}{2m}\,\frac{1}{4\alpha} = \frac{\hbar^2\alpha^2}{2m}$$

$$\bar{E}(\alpha) = \frac{\hbar^2\alpha^2}{2m} + \frac{3k}{2\alpha}, \quad \frac{\partial \bar{E}}{\partial\alpha} = \frac{\cancel{2}\hbar^2\alpha}{\cancel{2}m} - \frac{3k}{2\alpha^2}\Big|_{\alpha=\alpha_0} = 0$$

$$\alpha_0 = \left(\frac{3km}{2\hbar^2}\right)^{1/3}, \quad \bar{E}(\alpha_0) = \min \bar{E}(\alpha)$$

$$\left(\frac{243}{16}\right)^{1/3} \approx 2.48 \qquad = \left(\frac{3}{2}\right)^{5/3}\left(\frac{\hbar^2k^2}{2m}\right)^{1/3} 2^{1/3}$$

$$= \left(\frac{243}{16}\right)^{1/3}\left(\frac{\hbar^2 k^2}{2m}\right)^{1/3}$$

Problem 6 (continued)

(b) $$1 = B^2\int_0^\infty x^2 e^{-\alpha x^2}\,dx = -B^2\frac{\partial}{\partial\alpha}\int_0^\infty e^{-\alpha x^2}\,dx$$

$$= -B^2\frac{\partial}{\partial\alpha}\frac{1}{\sqrt{\alpha}}\int_0^\infty e^{-y^2}\,dy = \frac{B^2}{2\alpha^{3/2}}\frac{\sqrt{\pi}}{2} = \frac{B^2}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\bar{U} = kB^2\int_0^\infty x^3 e^{-\alpha x^2} = -kB^2\frac{\partial}{\partial\alpha}\int_0^\infty x e^{-\alpha x^2}\,dx$$

$$= -\frac{kB^2}{2}\frac{\partial}{\partial\alpha}\int_0^\infty dy\, e^{-\alpha y} = -\frac{kB^2}{2}\frac{\partial}{\partial\alpha}\frac{1}{\alpha} = \frac{kB^2}{2\alpha^2}$$

$$= \frac{2k}{\sqrt{\pi\alpha}}$$

$$\bar{T} = \frac{\hbar^2B^2}{2m}\int_0^\infty (1-\alpha x^2)^2 e^{-\alpha x^2}\,dx$$

$$= \frac{3\hbar^2\alpha}{4m}$$

$$\frac{\partial\bar{E}}{\partial\alpha} = \frac{3\hbar^2}{4m} - \frac{k}{\sqrt{\pi}\,\alpha^{3/2}}\bigg|_{\alpha=\alpha_0} = 0,\quad \alpha_0 = \left(\frac{4mk}{3\hbar^2}\right)^{2/3}\frac{1}{\pi^{1/3}}$$

$$\bar{E}(\alpha_0) = \min\bar{E} = \left(\frac{81}{2\pi}\right)^{1/3}\left(\frac{\hbar^2k^2}{2m}\right)^{1/3}$$

$$\left(\frac{81}{2\pi}\right)^{1/3} \approx 2.35$$

Compare with exact value in Problem 5: $\alpha_1 \approx 2.34$

2.31

Problem 7

$\psi(x) = A x e^{-\alpha|x|}$ is odd and therefore orthogonal do the g.s. wf

$$\overline{T} = \frac{\hbar^2}{2m}\int (\psi'(x))^2 dx = \frac{\hbar^2\alpha^2}{2m}$$

$$\overline{U} = \frac{3k}{2\alpha^2}$$

$$\overline{E}(\alpha) = \frac{\hbar^2\alpha^2}{2m} + \frac{3k}{2\alpha^2}, \quad \frac{\partial \overline{E}}{\partial \alpha} = \frac{\hbar^2\alpha}{m} - \frac{3k}{\alpha^3}\Big|_{\alpha_0} = 0$$

$$\alpha_0 = \left(\frac{3km}{\hbar^2}\right)^{1/}$$ [redacted]

$$E(\alpha_0) = \min E(\alpha) = \sqrt{3}\,\hbar\omega, \quad \omega = \sqrt{\frac{k}{m}}$$

Compare with $\frac{3}{2}\hbar\omega$ - exact energy of 1^{st} excited state

2.36

Problem 8

$$G_E(x,x') = \begin{cases} A(x')\, e^{\varkappa(x-x')} & , x < x' \\ B(x')\, e^{-\varkappa(x-x')} & , x > x' \end{cases}$$

$$\begin{cases} G_E'(x = x'+0, x') - G_E'(x = x'-0, x') = -\dfrac{2m}{\hbar^2} \\ G_E(x = x'+0, x') - G_E(x = x'-0, x') = 0 \end{cases}$$

$$A = B = \frac{m}{\varkappa \hbar^2}, \quad G_E(x,x') = \frac{m}{\varkappa \hbar^2} e^{-\varkappa|x-x'|}$$

Using $G_E(x,x')$, ~~[illegible]~~

$$-\frac{\hbar^2}{2m}\psi'' - E\psi = -U(x)\psi$$

Can be written for $E < 0$ as

$$\psi = -\frac{m}{\varkappa \hbar} \int_{-\infty}^{\infty} e^{-\varkappa|x-x'|}\, U(x')\psi(x')\,dx'$$

2.37

Problem 9

For $U(x) = -\alpha\delta(x)$, the integral equation in the previous problem takes the form

$$\psi(x) = \frac{\alpha m}{\varkappa\hbar^2}\psi(0)e^{-\varkappa|x|}$$

Substituting $x=0$,

$$\frac{\alpha m}{\varkappa\hbar^2} = 1 \quad \text{and} \quad E_0 = -\frac{\hbar^2\varkappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

2.41

Problem 10

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}G_E - EG_E = \delta(x-x')$$

$$G_E(x=0,x') = G_E(x=a,x') = 0$$

$E = \frac{\hbar^2\varkappa^2}{2m}$ $\Rightarrow$

$$G_E(x,x') = \begin{cases} A(x')\sin\varkappa x, & 0 \le x < x' \\ B(x')\sin\varkappa(x-a), & x' < x \le a \end{cases}$$

Matching at $x = x'$

$$G_E = \frac{-2m\sin[(x+x'-|x'-x|)\varkappa/2]\sin[(x+x'+|x'-x|-2a)\varkappa/2]}{\varkappa\hbar^2\sin\varkappa a}$$

$E_n = \frac{\hbar^2\varkappa_n^2}{2m}$, $\varkappa_n a = (n+1)\pi$ poles give energy levels in the well

2.43

Problem 11

$$\psi_k(x) = A(k) \sin kx \ , \quad k = \sqrt{\frac{2mE}{\hbar^2}} > 0$$

Orthonormality

$$\int_0^\infty \psi_k(x)\, \psi_{k'}(x)\, dx = \delta(k-k')$$

$$\parallel$$

$$\frac{1}{2} A(k) A(k') \int_0^\infty \left(\cos(k-k')x - \cos(k+k')x \right) dx$$

$$= \frac{\pi}{2} A(k) A(k') \delta(k-k') = \frac{\pi}{2} A^2(k)\, \delta(k-k')$$

$$\left(\text{using } \frac{1}{\pi} \int_0^\infty \cos kx\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx = \delta(k) \right)$$

$$\Rightarrow A(k) = \sqrt{\frac{2}{\pi}} \quad \text{and} \quad \psi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx$$

$$x > 0$$

$$\psi_E = \sqrt{\frac{dk}{dE}}\, \psi_k(x) = \left(\frac{2m}{\pi^2 E \hbar^2} \right)^{1/4} \sin \sqrt{\frac{2mE}{\hbar^2}}\, x$$

$$\int_0^\infty \psi_k(x)\, \psi_k(x')\, dk = \int_0^\infty \psi_E(x) \psi_E(x')\, dE = \delta(x-x')$$

2.50

Problem 12

$$\psi_k(x) = \begin{cases} e^{ikx}, & x<0 \\ A\sin kx + B\cos kx, & 0<x<a \\ Ce^{ik(x-a)}, & x>a \end{cases} \quad \left(k=\sqrt{\frac{2mE}{\hbar^2}}>0\right)$$

Matching at $x=0$ and $x=a$

$$\begin{cases} B=1 \\ kA - ik = \frac{2m\alpha}{\hbar^2} \end{cases} \quad (\text{at } x=0)$$

$$\begin{cases} A\sin ka + B\cos ka = C \\ ikC - kA\cos ka + kB\sin ka = \frac{2m\alpha C}{\hbar^2} \end{cases} \quad (\text{at } x=a)$$

4 equations, 3 variables (A, B, C) means that can be satisfied only for certain values of k. Eliminating $A, B,$ and C

$$\tan ka = -\frac{\hbar^2 k}{\alpha m}$$

determines energies $E = \frac{\hbar^2 k^2}{2m}$ for which there is no reflection