Assignment #1 – Solutions

(1.1) Problem 1

$\hat{I}, \hat{T}_a, \hat{M}_c$ are linear, $\hat{K}$ is not

$$\int_{-\infty}^{\infty} \psi^*(x)\, \hat{M}_c\, \phi(x) \equiv \int \psi^*(x) \sqrt{c}\, \phi(cx)\, dx$$

$$= \frac{1}{\sqrt{c}} \int_{-\infty}^{\infty} \psi^*\left(\frac{x}{c}\right) \phi(x)\, dx \equiv \int \left(\hat{\tilde{M}}_c\, \psi^*(x)\right) \phi(x)\, dx$$

$$\Rightarrow \hat{\tilde{M}}_c\, \psi^*(x) \equiv \frac{1}{\sqrt{c}}\, \psi^*\left(\frac{x}{c}\right) = \hat{M}_{1/c}\, \psi^*(x)$$

$$\hat{\tilde{M}}_c = \hat{M}_{1/c}$$

$$\hat{M}_c^{\dagger} \equiv \hat{\tilde{M}}_c^{*} = \hat{M}_{1/c}$$

Also
$$\hat{I}^{\dagger} = \hat{\tilde{I}} = \hat{I}$$
$$\hat{\tilde{T}}_a = \hat{T}_a^{\dagger} = \hat{T}_{-a}$$

Inverse operators

$$\hat{I}^{-1} = \hat{I}, \quad \hat{K}^{-1} = \hat{K}$$
$$\hat{T}_a^{-1} = \hat{T}_{-a}, \quad \hat{M}_c^{-1} = \hat{M}_{1/c}$$

(1.12) Problem 2

a) Since $\hat{I}^2 = 1$,

$$e^{i\pi\hat{I}} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\pi\hat{I})^n = \cos\pi + i\hat{I}\sin\pi = 1$$

b) $$\hat{T}_a \psi(x) = \sum_n \frac{a^n}{n!}\left(\frac{d}{dx}\right)^n \psi(x)$$

$$= \sum_n \frac{a^n}{n!}\psi^{(n)}(x) = \psi(x+a)$$

(1.13) Problem 3

$$(\hat{A} - \lambda\hat{B})^{-1} = \sum_{n=0} \lambda^n \hat{C}_n$$

$$1 = (\hat{A} - \lambda\hat{B})\sum_n \lambda^n \hat{C}_n$$

Equating terms with the same powers of λ

$$\hat{A}\hat{C}_{n+1} = \hat{B}\hat{C}_n, \quad \hat{C}_{n+1} = \hat{A}^{-1}\hat{B}\hat{C}_n \text{ and } \hat{C}_0 = \hat{A}^{-1}$$

Therefore,

$$\frac{1}{\hat{A} - \lambda\hat{B}} = \hat{A}^{-1} + \lambda\hat{A}^{-1}\hat{B}A^{-1} + \dots$$

$$= \hat{A}^{-1}\sum_{n=0}^{\infty} \lambda^n (\hat{B}\hat{A}^{-1})^n$$

1.14 Problem 4

Consider $\hat{f}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$

$$\frac{d\hat{f}}{d\lambda} = \hat{A}e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} - e^{-\lambda\hat{A}}\hat{B}\hat{A}e^{-\lambda\hat{A}}$$

$$= e^{\lambda\hat{A}}[\hat{A},\hat{B}]e^{-\lambda\hat{A}}$$

Analagously,

$$\frac{d^2\hat{f}}{d\lambda^2} = e^{\lambda\hat{A}}[\hat{A},[\hat{A},\hat{B}]]e^{-\lambda\hat{A}}, \text{ etc.}$$

Finally,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{f}(\lambda=1) = \sum_n \frac{1}{n!}\left(\frac{d^n f}{d\lambda^n}\right)_{\lambda=0}$$

$$= \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

1.15 Problems

$$L^*(\xi,\xi') = (L(\xi,\xi'))^*$$

$$\tilde{L}(\xi,\xi') = L(\xi',\xi)$$

$$L^\dagger(\xi,\xi') = L^*(\xi',\xi)$$

$$\hat{M}_c \psi(x) = \sqrt{c}\,\psi(cx) = \sqrt{c}\int_{-\infty} \delta(cx-x')\,\psi(x')\,d$$

$$\Rightarrow M_c(x,x') = \sqrt{c}\,\delta(cx-x') = \frac{1}{\sqrt{c}}\,\delta\left(x-\frac{x'}{c}\right)$$

Similarly

$$I(x,x') = \delta(x+x')$$

$$T_a(x,x') = \delta(x-x'+a)$$

$$X(x,x') = x\,\delta(x-x')$$

$$P(x,x') = -i\hbar\frac{\partial}{\partial x}\delta(x-x')$$

Notice: $\frac{\partial}{\partial x}\delta(x-x') = -\frac{\partial}{\partial x'}\delta(x-x')$

$$\hat{p}\,\psi(x) = \int dx'\,\psi(x')\left[i\hbar\frac{\partial}{\partial x'}\delta(x-x')\right]$$

$$= -i\hbar\int dx'\left[\frac{\partial}{\partial x'}\psi(x')\right]\delta(x-x')$$

1.17 Problem 6

For $\hat{C} = \hat{A}\hat{B}$,

$$C(x,x') = \int A(x,x'')B(x'',x')dx''$$

For $\hat{x}$,

$$X(x,x') = x\delta(x-x')$$

From $\hat{F}\hat{x} - \hat{x}\hat{F} = 0$

$$\int \left[F(x,x'')x''\delta(x''-x') - x\delta(x-x'')F(x'',x')\right]$$
$$= (x'-x)F(x,x') = 0$$

Consequently,

$$F(x,x') = f(x)\delta(x-x')$$

where $f(x)$ is an arbitrary fn.

Similarly, from $\hat{G}\hat{p} - \hat{p}\hat{G} = 0$ and

$$P(x,x') = -i\hbar\frac{\partial}{\partial x}\delta(x-x'),$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)G(x,x') = 0,$$

(1.19) Problem 7

From $\int dx\, |\psi(x)|^2 = 1$, $|c| = (\pi a^2)^{-1/4}$

$$\bar{x} = x_0, \quad \overline{x^2} = \frac{a^2}{2} + x_0^2, \quad \overline{(\Delta x)^2} = \overline{x^2} - \bar{x}^2 =$$

$$\bar{p} = p_0, \quad \overline{p^2} = p_0^2 + \frac{\hbar^2}{2a^2}, \quad \overline{(\Delta p)^2} = \overline{p^2} - \bar{p}^2 =$$

(1.25) Problem 8

Eq. for EV and EF is

$$-i\tilde{\alpha} \frac{d}{dx} \psi_f(x) + \beta x\, \psi_f(x) = f\, \psi_f(x)$$

whose solution is

$$\psi_f(x) = C \exp\left\{ -\frac{i(\beta x - f)^2}{2\tilde{\alpha}\beta} \right\}$$

Here $\tilde{\alpha} = \alpha\hbar$.

Continuos spectrum with arbitrary real

Normalizing $\int \psi^*_{f'}(x)\, \psi_f(x) = \delta(f - f')$

yields $|c| = (2\pi\tilde{\alpha})^{-1/2}$