QM 15-030-710-00S Summer ****
Assignment S: Scattering

The due date for this assignment is **.**
Reading assignment: Baym - Chapters 6, 9, and 10; LL - Chapters V, $XVII$, and $XVIII$.

1. Show that the scattering amplitude in an arbitrary external field can be expressed in terms of the WF within the effective range of the potential as

$$f(\mathbf{k},\mathbf{k}_0) = -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r})\, U(\mathbf{r})\, \psi^{(+)}_{\mathbf{k}_0}(\mathbf{r})\, dV$$

   where $\mathbf{k}_0$, $\mathbf{k}$ are the wave vectors of the particle before and after scattering and $\psi^{(+)}_{\mathbf{k}_0}$ is the WF whose asymptotic behavior at $r \to \infty$ is

$$\psi^{(+)}_{\mathbf{k}_0}(\mathbf{r}) \approx \exp(i\mathbf{k}_0\cdot\mathbf{r}) + f(\mathbf{k},\mathbf{k}_0)\frac{\exp(ikr)}{r}$$

   Use your result to derive the expression for the scattering amplitude in the Born approximation.

   Hint: Use the Green's function to rewrite the Schrödinger equation in the integral form and expand at $r \to \infty$.

2. How fast should the potential decay at large distances, $r \to \infty$, for the asymptotic form of the WF $\psi^{(+)}_{\mathbf{k}_0}(\mathbf{r})$ at $r \to \infty$ to be as in the preceding problem (that is, for the plane wave not to be distorted at large distances by the potential). Limit your consideration to potentials whose asymptotic behavior at $r \to \infty$ is $U(r) \sim r^{-n}$.

   Hint: Consider convergence of the integral $\int^{\infty} r^{-n+1}\sin(qr)\, dr$, where $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$.

3. In the Born approximation, find the scattering amplitude and the cross section for the following potentials:

   a)
$$U(r) = \alpha\delta(r - R)$$

   b)
$$U(r) = U_0 \exp(-r/R)$$

   c)
$$U(r) = (\alpha/r)\exp(-r/R)$$

   d)
$$U(r) = \alpha/r^2$$

   e)
$$U(r) = \begin{Bmatrix} U_0, \; r < R \\ 0, \; r > R \end{Bmatrix}$$

   f)
$$U(r) = U_0 \exp\left(-r^2/R^2\right)$$

   Consider limiting cases of large and small energies and determine the conditions of applicability of the approximation.

4. An important quantity considered in investigation of scattering processes is

$$\eta(E) = \frac{\mathrm{Re}\left(f(E,\theta=0)\right)}{\mathrm{Im}\left(f(E,\theta=0)\right)}$$

   a) Show that $\eta(E)$ has a definite sign, given whether the potential is attractive or repulsive;

   b) Express $|\eta(E)|$ in terms of the differential and total cross-section of forward scattering;

   c) Show that $|\eta(E)| \gg 1$ except in the case of oscillating potential.

5. In the Born approximation, find the scattering amplitude in a double-potential

$$U(\mathbf{r}) = U_0(\mathbf{r}) + U_0(\mathbf{r} - \mathbf{a})$$

in terms of the scattering amplitude f_0^B of a single potential $U_0(\mathbf{r})$. Find the relationship between the cross-section of the double-potential and the cross-section σ_0 of a single potential in the limiting cases of $ka \ll 1$ (in which case kR is arbitrary, where R is the range of U_0) and $kR \gtrsim 1$ and $a \gg R$ (in which case the distance between the centers is much larger than the range of U_0).

6. Restore the potential $U(r)$ from the phase shift $\delta_0(k)$ $(l = 0)$, known for all energies, assuming that $|\delta_0(k)| \ll 1$. Illustrate your answer for the following examples:

   a)
   $$\delta_0(k) = const$$

   b)
   $$\delta_0(k) = \frac{\alpha k}{1 + \beta k^2}$$

   Hint: Born approximation is sufficient for $|\delta_0(k)| \ll 1$. Show that

   $$-\frac{\hbar^2}{2m}\frac{d}{dk}(k\delta_0(k)) = \frac{i}{2}\int_{-\infty}^{\infty} rU(r)\exp(-2ikr)\,dr$$

   and use inverse FT.

7. Find the exact expressions for the s-wave phase shift in the following potentials:

   a)
   $$U(r) = \begin{Bmatrix} \infty, \; r < R \\ 0, \; r > R \end{Bmatrix}$$

   b)
   $$U(r) = \begin{Bmatrix} -U_0, \; r < R \\ 0, \; r > R \end{Bmatrix}$$