QM 15-030-710-002 Winter **
Assignment 4: Motion in Central Field**

The due date for this assignment is **.**
Reading assignment: Chapter V.

1. Find the energy levels and the normalized wave functions of the stationary states of a plane harmonic oscillator $U(\rho) = k\rho^2/2$ and determine the degeneracy of such states.

   Hint: Use separation of variables in Cartesian coordinates and label the states $\psi_{n_1 n_2}$, where n_1 and n_2 are the quantum numbers of 1D oscillators.

2. In a stationary state ψ_{11} (see preceding problem) of a plane harmonic oscillator, find the probabilities of various possible projections of the angular momentum along the axis perpendicular to the plane of oscillations.

   Hint: Express the WF in polar coordinates and use $\sin(2\phi) = \left(e^{2i\phi} - e^{-2i\phi}\right)/2i$.

3. Find the energies of the discrete spectrum levels in a 2D field $U(\rho) = -\alpha/\rho$ and their degeneracy. Compare with the case of the Coulomb field $U(r) = -\alpha/r$.

4. Find an approximate energy of the ground state of a 2D oscillator using the trial function $\psi(\rho) = C\exp(-\alpha\rho)$ and the variational method.

5. Find the energy levels and the normalized wave functions of the stationary states of a spherical harmonic oscillator $U(r) = kr^2/2$ and determine the degeneracy of such states.

6. Find the effective (mean) potential $\varphi(r)$ acting on a charged particle in a hydrogen atom in the ground state, neglecting the polarization of the latter.

   Hint: Use
   $$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$
   where $r_<$ and $r_>$ denote the lesser and the greater, respectively, of r and r', and the orthogonality of Legendre polynomials.

7. Find an approximate energy of the ground state of a particle in the Coulomb field $U(r) = -\alpha/r$ using the trial function $\psi(r) = C\exp\left(-\varkappa^2 r^2\right)$ and the variational method. Compare with the exact result.

8. Find the Green's function $G_E(\mathbf{r}, \mathbf{r}')$ of the Schrödinger's equation for a free particle with $E < 0$,
   $$\left(\widehat{H} - E\right) G_E \equiv -\frac{\hbar^2}{2m}\nabla^2 G_E - EG_E = \delta(\mathbf{r} - \mathbf{r}')$$
   such that it decays when $r \to \infty$. Use the latter to derive an integral form of the Schrödinger's equation for discrete spectrum states of a particle in the field $U(r)$ that decays for $r \to \infty$.