QM 15-030-710-001 Fall **
Assignment 3: Angular momentum

The due date for this assignment is **.**
Reading assignment: Chapter IV.

1. Show that the equality $\mathbf{l}^2 = l\,(l+1)$ follows an assumption that all possible values, $m = -l$, $-l+1$, ..., l, of the projection of the angular momentum along an arbitrary axis are equally probable, taking into account that all such axes are equivalent.

   Hint: $\sum_{m=0}^{l} m^2 = \frac{d^2}{d\alpha^2}\left[\sum_{m=0}^{l} e^{\alpha m}\right]_{\alpha=0}$

2. Show that the functions obtained by application of the operators $\widehat{l}_{\pm} = \widehat{l}_x \pm i\widehat{l}_y$ to the eigenfunctions Ψ_m of $\widehat{l}_z$, $\widehat{l}_z\Psi_m = m\Psi_m$, will also be eigenfunctions whose eigenvalues are $m \pm 1$.

3. Prove the following relationships for a state Ψ_m corresponding to the quantum number m for the projection of the angular momentum along the z-axis:

   - $\overline{l_x} = \overline{l_y} = 0$;
   - $\overline{l_x l_y} = -\overline{l_y l_x} = im/2$;
   - $\overline{\widehat{l}_x^2} = \overline{\widehat{l}_y^2}$.

4. In a state Ψ_{lm} with the quantum numbers l and m for the angular momentum and its projection along the z-axis, find the mean value and the variance of the fluctuation of the projection of the angular momentum along the axis $\widetilde{z}$ directed at an angle α with respect to z.

5. For a particle with the angular momentum $l = 1$, find the angular dependence of the wavefunction $\Psi_{\underset{\sim}{m}}(\theta, \phi)$, θ, ϕ being the angular variables in the coordinate system with the polar axis z, of a state with a zero projection, $\widetilde{m} = 0$, of the angular momentum along a $\widetilde{z}$-axis whose direction is defined, respectively, by the polar and azimuthal angles α and β.

6. For a particle state with the angular momentum $l = 1$ and its projection m along the z-axis, find $\overline{\widehat{l}_x^n}$ and $\overline{\widehat{l}_y^n}$ where n is integer.

7. Find the spectrum of a squared vector product of the two angular momenta, $(\mathbf{l}_1 \times \mathbf{l}_2)^2$.

   Hint: $\varepsilon_{ikl}\varepsilon_{\alpha\beta l} = \delta_{i\alpha}\delta_{k\beta} - \delta_{i\beta}\delta_{k\alpha}$

8. Consider two weakly interacting systems with quantum numbers (l_1, m_1) and (l_2, m_2).

   - Find the possible values of the total moment L of the combined system $(1+2)$ and evaluate $\overline{\widehat{\mathbf{L}}}$ and $\overline{\widehat{\mathbf{L}}^2}$;
   - For a particular case $m_1 = l_1$ and $m_2 = l_2 - 1$, find the probabilities of the possible values of L.