QM 15-030-710-001 Fall ****
Assignment 2: One-dimensional motion

The due date for this assignment is **.**
Reading assignment: Chapter *III*.

1. The state of a particle in an infinite well of width a $(0 < x < a)$ is described by the wave function (WF) of the form

   (a) $\Psi(x) = Ax(x-a)$
   (b) $\Psi(x) = B\sin^2\left(\frac{\pi x}{a}\right)$

   Find the following:

   - distribution of energy eigenvalues (EV)
   - mean value of energy
   - mean square fluctuation of energy

2. Find even and odd discrete energy levels of a particle in the potential well

$$U(x) = \begin{matrix} -U_0, \, |x| < a \\ 0, \quad |x| > a \end{matrix}$$

   - What is the number of discrete state as a function of the depth U_0 of the well?
   - What is the condition for appearance of the new levels of discrete spectrum as the well deepens?
   - Find the lowest energy levels for a deep well, $U_0 \gg \frac{\hbar^2}{2ma^2}$, and compare these with the infinitely deep well.

3. Find the energy levels and the normalized WFs of the discrete spectrum states of a particle in the well

$$U(x) = -\alpha\delta(x), \; \alpha > 0$$

   Find the mean values of the kinetic and potential energies in these states.

4. For a particle in the stationary states of the field

$$U(x) = U_0 f\left(\frac{x}{a}\right)$$

   find the dependence of the energy levels E_n and the mean values $\overline{x}$ and $\overline{(\Delta x)^2}$ on parameters U_0 or a given that $\frac{ma^2U_0}{\hbar^2} = \text{const}$.

5. Find the WFs of stationary states and the energy levels of a particle in the uniform gravity field g if the motion of the particle is limited from below by the ideally reflective plane.

6. For a particle in the field

$$U(x) = \begin{matrix} kx, \, x > 0 \\ \infty, \, x < 0 \end{matrix} \quad (k > 0)$$

   find the variational ground state using the following trial functions $(x > 0)$:

   (a) $\Psi(x) = Ax\exp(-\alpha x)$
   (b) $\Psi(x) = Bx\exp\left(-\frac{\alpha x^2}{2}\right)$

   where α is the variational parameter. Compare your result with the exact result of Problem 5.

7. Using $\Psi(x) = Ax\exp(-\alpha|x|)$ as trial function (α is the variational parameter), find the energy of the first excited state of a harmonic oscillator and compare with the exact result.

8. Find the Green's function (GF) $G_E(x, x')$ of the Schrödinger's equation for a free particle with $E < 0$ such that it decays when $|x - x'| \to \infty$. The Green's function satisfies the following equation:

$$\left(\widehat{H} - E\right) G_E \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} G_E - EG_E = \delta(x - x')$$

9. Find the ground state energy E_0 and the normalized WF of the ground state $\Psi_0(x)$ of a particle in the field

$$U(x) = -\alpha\delta(x)\,,\ \alpha > 0$$

using the integral form of the Schrödinger's equation and the preceding problem. Compare with Problem 3.

10. Find the GF $G_E(x, x')$ for a particle in an infinitely deep well of width a $(0 < x < a)$. Discuss the analytical properties of G_E as a function of E. Show, in particular, that it has poles and establish a correspondence between the position of the poles in the complex plane of variable E and the energy levels E_n of the particle.

11. For a free particle in the field

$$U(x) = \begin{matrix} \infty,\, x < 0 \\ 0,\ x > 0 \end{matrix}$$

find the WFs of the stationary states and normalize them to the energy δ-function,

$$\int_0^\infty \Psi_E(x)\,\Psi_{E'}^*(x) = \delta(E - E')$$

Show that such WFs form a complete set for $x > 0$.

12. Find the energy levels for which a particle would not reflect from the potential barrier of the form

$$U(x) = \alpha\left[\delta(x) + \delta(x - a)\right],\ \alpha > 0$$