QM 15-030-710-001 Fall ****
Assignment 1: Operators in quantum mechanics

The due date for this assignment is **.**
Reading assignment: Chapters I and II.

1. Consider the following operators ($-\infty < x < \infty$):

   (a) reflection, $\widehat{I}\Psi(x) \equiv \Psi(-x)$

   (b) translation, $\widehat{T}\Psi(x) \equiv \Psi(x+a)$

   (c) scale transformation, $\widehat{M_c}\Psi(x) \equiv \sqrt{c}\Psi(cx)$

   (d) complex conjugation, $\widehat{K}\Psi(x) \equiv \Psi^*(x)$

   Are these operators linear? Find operators which, with respect to these operators, are

   - transposed
   - complex-conjugate
   - Hermitian conjugate

2. An operator $\widehat{F} = F\left(\widehat{f}\right)$, where $F(z)$ is a function that can be represented as a series $F(z) = \sum_n c_n z^n$, can be understood as an operator such that $\widehat{F} \equiv \sum_n c_n \widehat{f}^n$. Using this definition, find the explicit form of the following operators:

   (a) $\exp\left(i\pi\widehat{I}\right)$

   (b) $\widehat{T}_a = \exp\left(a\frac{d}{dx}\right)$

   where operator $\widehat{I}$ is defined in Problem 1.

3. Assuming that λ is a small quantity, find the expansion of the operator $\left(\widehat{A} - \lambda\widehat{B}\right)^{-1}$ in powers of λ.

4. Prove the following relationship:

$$\exp\left(\widehat{A}\right)\widehat{B}\exp\left(-\widehat{A}\right) = \widehat{B} + \left[\widehat{A},\widehat{B}\right] + \frac{1}{2!}\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right] + \ldots$$

5. Generally, a linear operator $\widehat{L}$ can be considered as a linear integral operator, i.e.

$$\Phi(\xi) \equiv \widehat{L}\Psi(\xi) \equiv \int L\left(\xi,\xi^{'}\right)\Psi\left(\xi^{'}\right)d\xi^{'}$$

   where $L\left(\xi,\xi^{'}\right)$ is the kernel of the operator $\widehat{L}$ and ξ are the variables of this representation.. How are the kernel of the operators $\widehat{L}^*$, $\widetilde{\widehat{L}}$, $\widehat{L}^\dagger$ related to $L\left(\xi,\xi^{'}\right)$? Find the kernels of the following operators:

   (a) $\widehat{I}$

   (b) $\widehat{T}$

   (c) $\widehat{M_c}$

   (d) $\widehat{x}$

   (e) $\widehat{p} = -i\hbar\frac{d}{dx}$

   where the former three are defined in Problem 1.

6. What is the form of the kernel $L\left(x,x^{'}\right)$ of an operator $\widehat{L}$ which commutes with the operators of:

   (a) coordinate $\widehat{x} \equiv x$

(b) momentum $\widehat{p} \equiv -i\hbar \frac{d}{dx}$

7. In a state described by the wave function (WF) of the form

$$\Psi(x) = C \exp\left[\frac{ip_0 x}{\hbar} - \frac{(x - x_0)^2}{2a^2}\right]$$

where p_0, x_0, and a are real, find the coordinate distribution function. Determine the mean values and the fluctuations of the coordinate and momentum.

8. Find eigenvalues (EV) and eigenfunctions (EF) of a physical quantity which is a linear combination of the coordinate and momentum, $f = \alpha\widehat{x} + \beta\widehat{p}$. Prove the orthogonality of such EFs and normalize them appropriately. Consider the limiting cases of $\alpha \to 0$ and $\beta \to 0$.