QM 15-Phys-710-001(/002/003) Winter 2001
Midterm: Angular Momentum. Central Field.
Wednesday, February 7

1. A *rotator* is a rotating system of two rigidly connected particles with the moment of inertia I ($I = \mu a^2$, where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the particles and a is the distance between them). A state of the spatial rotator is described by the wave function

$$\psi = C \sin^2\left(\frac{\theta}{2}\right)$$

(a) Normalize the wave function.

(b) Find the expectation value of energy.

Hint:

$$Y_{00} = \sqrt{\frac{1}{4\pi}},\ Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Solution

Expectation value of energy

$$\psi = C\sin^2\left(\frac{\theta}{2}\right) = C\frac{1-\cos\theta}{2} = C\sqrt{\pi}\left(Y_{00} - \frac{Y_{10}}{\sqrt{3}}\right)$$

$$w(0) + w(1) = 1,\ \frac{w(0)}{w(1)} = 3$$

$$w(0) = \frac{3}{4},\ w(1) = \frac{1}{4}$$

$$E = \frac{\hbar^2}{2I}\overline{\hat{\mathbf{l}}^2} = \sum_l \frac{\hbar^2}{2I} w(l)\left[l(l+1)\right] = \frac{\hbar^2}{I}w(1) = \frac{\hbar^2}{4I}$$

Normalization

$$2\pi\int_0^\pi \psi^2 \sin(\theta)\, d\theta = 1$$

$$\begin{aligned} \frac{\pi C^2}{2}\int_{-1}^{1}(1-x)^2\,dx &= \frac{\pi C^2}{2}\frac{8}{3} = 1 \\ C &= \sqrt{\frac{3}{4\pi}} \end{aligned}$$

2. The equation for the radial part of the wave function in the attractive Coulomb potential is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + \frac{2m}{\hbar^2}\left(E + \frac{e^2}{r}\right)R = 0$$

Find the expectation value of the potential and kinetic energies in the ground state.

Solution

In atomic units,

$$R_0 = 2\exp(-r)$$

The expectation value of the potential energy is

$$\overline{U} = -\overline{\frac{1}{r}} = -\int_0^\infty R_0^2\frac{1}{r}r^2dr = -4\int_0^\infty \exp(-2r)\,r\,dr = -1$$

The total energy of the ground state is

$$\begin{aligned} E &= \overline{U} + \overline{K} = -\frac{1}{2} \\ \overline{K} &= \frac{1}{2} \end{aligned}$$