QM 15-Phys-710-001(/002/003) Winter 2001
Final
Monday, March 12

1. (20 POINTS) A uniform electric field is suddenly applied to a charged one-dimensional harmonic oscillator in the ground state.

   a) Evaluate the probability that the oscillator will remain in the ground state. Comment on your answer in the limiting cases of weak and strong fields.

   b) Evaluate the probability of transition to the first excited states. Comment on your answer in the weak field limit.

   Solution

   In dimensionless variables

$$\begin{aligned}\psi_0(x) &= \pi^{-1/4}\exp\left(-x^2/2\right)\\ \psi_1(x) &= 2^{1/2}\pi^{-1/4}x\exp\left(-x^2/2\right)\end{aligned}$$

   After the perturbation, the WF's are $\psi_0(x-x_0)$ and $\psi_1(x-x_0)$, where

$$x_0 = \frac{eE/m\omega^2}{\sqrt{\hbar/m\omega}}$$

   The probabilities are

$$w_{00} = \left[\int \psi_0(x)\,\psi_0(x-x_0)\,dx\right]^2 = \exp\left(-x_0^2/2\right)$$

   and

$$w_{01} = \left[\int \psi_0(x)\,\psi_1(x-x_0)\,dx\right]^2 = \left(x_0^2/2\right)\exp\left(-x_0^2/2\right)$$

   Limits: $w_{00}\to 1$ when $x_0 \ll 1$ (week field) and $w_{00}\to 0$ when $x_0 \gg 1$ (strong field); $w_{01}\to x_0^2/2 \ll 1$ when $x_0 \ll 1$.

2. (20 POINTS) Consider a particle in the Coulomb field

$$U = -\frac{\alpha}{r}$$

   In the limit of a large angular momentum $l \gg 1$ and $l \gg n_r$, where n_r is the radial quantum number and $n = n_r + l + 1$ is the principal quantum number, the energy levels can be found as those of a harmonic oscillator that approximates the effective potential

$$U_{eff} = -\frac{\alpha}{r} + \frac{\hbar^2 l(l+1)}{2mr^2}$$

   near the minimum. Show that this result is consistent with the exact energy levels of a Coulomb field in the considered limit.

   Hint: $l(l+1) \approx \left(l+\frac{1}{2}\right)^2$.

   Solution

   The minimum of U_{eff} is at

$$r_0 = \frac{\hbar^2 l(l+1)}{m\alpha} \approx \frac{\hbar^2\left(l+\frac{1}{2}\right)^2}{m\alpha}$$

   whereof the frequency of the harmonic oscillator

$$\omega = \sqrt{\frac{U''_{eff}(r_0)}{m}} = \frac{m\alpha^2}{\left[\hbar^2 l(l+1)\right]^{3/2}} \approx \frac{m\alpha^2}{\hbar^3\left(l+\frac{1}{2}\right)^3}$$

   and the energies are

$$E_{n_r l} = U_{eff}(r_0) + \hbar\omega\left(n_r + \frac{1}{2}\right) \approx -\frac{m\alpha^2}{2\hbar^2\left(l+\frac{1}{2}\right)^2} + \frac{m\alpha^2}{\hbar^2\left(l+\frac{1}{2}\right)^3}\left(n_r+\frac{1}{2}\right)$$

Using the exact energies,

$$\begin{aligned} E_n &= -\frac{m\alpha^2}{2\hbar^2 n^2} = -\frac{m\alpha^2}{2\hbar^2 \left(n_r + l + 1\right)^2} = -\frac{m\alpha^2}{2\hbar^2 \left(n_r + \frac{1}{2} + l + \frac{1}{2}\right)^2} \\ &\approx -\frac{m\alpha^2}{2\hbar^2 \left(l + \frac{1}{2}\right)^2} + -\frac{m\alpha^2}{\hbar^2 \left(l + \frac{1}{2}\right)^3} \left(n_r + \frac{1}{2}\right) \end{aligned}$$

3. (20 POINTS) For a planar (two-dimensional) harmonic oscillator subject to the perturbation $V = \alpha xy$

   a) Find the energy splitting of the first excited state.

   b) Find the correct unperturbed eigenfunctions.

   c) (bonus) The problem can be actually solved exactly by a proper rotation of the coordinate system. Compare the exact energy eigenvalues with the ones obtained using the perturbation theory.

   Solution

   The level is doubly degenerate with two WF's $\Psi_1 = \psi_0(x)\psi_1(y)$ and $\Psi_2 = \psi_0(y)\psi_1(x)$ having the same energy $2\hbar\omega$. The matrix elements are

$$\begin{aligned} V_{11} &= V_{22} = 0 \\ V_{12} &= V_{21} = \alpha a^2/2 \end{aligned}$$

   where $a = \sqrt{\hbar/m\omega}$. The secular equation

$$\begin{vmatrix} -E^{(1)} & \alpha a^2/2 \\ \alpha a^2/2 & -E^{(1)} \end{vmatrix} = 0$$

   and $E^{(1)} = \pm\alpha a^2/2$. The correct unperturbed WF's are

$$\Psi_{1,2}^{(0)} = \left(\Psi_1 \pm \Psi_2\right)/\sqrt{2}$$