QM 15-Phys-710
Fall 2000
Quiz 2: Energy and Momentum. Schrödinger Equation.
Monday, November 13

Please complete only **one** of the following two tests. A combination of parts will **not** be considered.

Test 1

1. Derive the stationary-state Schrödinger equation in the momentum representation for a particle in the potential $U(x)$. Use this equation to derive the energy level in the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$ (*Note*: solutions in coordinate representation will not be accepted).

   Solution

$$\frac{p^2}{2m}a(p) + \int_{-\infty}^{\infty} V\left(p - p'\right) a\left(p'\right) dp' = Ea(p)$$

where

$$V(p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} U(x)\exp\left(-\frac{ipx}{\hbar}\right)dx$$

and

$$a(p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \psi(x)\exp\left(-\frac{ipx}{\hbar}\right)dx$$

For $U(x) = -\alpha\delta(x)$,

$$V(p) = -\frac{\alpha}{2\pi\hbar}$$

and

$$\frac{p^2}{2m}a(p) - \frac{\alpha}{2\pi\hbar}\int_{-\infty}^{\infty} a(p)\,dp = Ea(p)$$

Denote $\int_{-\infty}^{\infty} a(p)\,dp = C$, then

$$a(p) = \frac{\alpha m}{\pi\hbar}\frac{C}{p^2 + 2m|E|}$$

Substitution back into $\int_{-\infty}^{\infty} a(p)\,dp = C$ gives

$$1 = \frac{\alpha}{\hbar}\sqrt{\frac{m}{2|E|}}$$

and

$$E_0 = -\frac{m\alpha^2}{2\hbar^2}$$

Test 2

1. For an infinite well of width a $(0 < x < a)$

   a) Find the normalized wave functions $\psi_n(x)$ of stationary states and their energies E_n (you have to *derive* your results).

   The state of a particle in the well is described by the wave function of the form $\psi(x) = Ax(x-a)(x-a/2)$.

   b) Normalize the wave function;

   c) Sketch $\psi(x)$ and compare it with $\psi_1(x)$, the first excited eigenstate;

   d) For this state, find the expectation value of

   - potential energy $\overline{U}$;
   - kinetic energy $\overline{T}$
   - total energy $\overline{E}$ and compare it with the energy of the first excited eigenstate. Explain your result.

1. *Hint*: in b) and d) it may be convenient to make a change of variable $y = x - a/2$.

   Solution

   a)

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin \frac{\pi(n+1)x}{a}, \ 0 < x < a \\ 0, \ x < 0 \text{ and } x > a \end{cases}$$

$$E_n = \frac{\hbar^2 \pi^2 (n+1)^2}{2ma^2}$$

   b) $A^2 = 840/a^7$

   d) $\overline{U} = 0$, $\overline{E} = \overline{T} = 21\hbar^2/ma^2$ and $\overline{E}/E_1 = 21/2\pi^2$ - larger than, but close to, 1. This is because $\psi(x)$ and $\psi_1(x)$ have a large overlap.

2. Consider the following potential:

$$U(x) = \begin{cases} U_0 > 0, \ x > 0 \\ 0, \ x < 0 \end{cases}$$

   a) For a particle with $E > U_0$, can the transmission coefficient be 1?

   b) What are the transmission and reflection coefficients for a particle whose energy is $E < U_0$?

   Solution

   a) Always less than 1; approaches 1 when $E \gg U_0$.

   b) Transmission and reflection coefficients are 0 and 1 respectively.