QM 15-Phys-710-001(/002/003) Winter 2000
Midterm: Angular Momentum. Central Field.
Monday, February 14

1. A *rotator* is a rotating system of two rigidly connected particles whose moment of inertia is $I = \mu a^2$, where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the particles and a is the distance between them. For the state of a spatial rotator described by the wave function

$$\psi = C \cos(2\theta)$$

find the distribution function of the squared of the angular momentum (that is, probabilities $w(l)$) and the expectation values of energy.

Hint:

$$Y_{00} = \sqrt{\frac{1}{4\pi}},\ Y_{20} = \sqrt{\frac{5}{16\pi}} \left(1 - 3\cos^2\theta\right)$$

Solution

Re-writing ψ as

$$\psi = C\left(2\cos^2\theta - 1\right) = \frac{\sqrt{4\pi}C}{3}\left[-\frac{1}{\sqrt{4\pi}} - \frac{2\left(1 - 3\cos^2\theta\right)}{\sqrt{4\pi}}\right] = \frac{\sqrt{4\pi}C}{3}\left[-Y_{00} - \frac{4}{\sqrt{5}}Y_{20}\right]$$

we find $m = 0$, and $l = 0$ and $l = 2$ as the only possible values of l whose probabilities are found from

$$w(0) + w(2) = 1 \text{ and } w(0)/w(2) = 5/16$$

whereof $w(0) = 5/21$ and $w(2) = 16/21$.

The energy eigenvalues of the spatial rotator are

$$E_l = \frac{\hbar^2 l(l+1)}{2I}$$

yielding the following expectation value of energy:

$$\overline{E} = \frac{\hbar^2}{2I}\sum_{l=0}^{\infty} w(l)\, l(l+1) = \frac{\hbar^2}{2I} w(2)\, 6 = \frac{16\hbar^2}{7I}$$

2. The equation for the radial part of the wave function in the attractive Coulomb potential is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + \frac{2m}{\hbar^2}\left(E + \frac{\alpha}{r}\right)R = 0$$

(a) Construct the units of energy and length from m, $\hbar$, and α (Coulomb units) and re-write the radial equation in the dimensionless form;

(b) Find the asymptotic behavior of R at large and small distances;

(c) Find the energy and the normalized radial function of the ground state.

Solution

(a) Units of length: $\ell = \hbar^2/m\alpha$, units of energy: $\varepsilon = m\alpha^2/\hbar^2$, as found from $\varepsilon = \alpha/\ell = \hbar^2/m\ell^2$. In Coulomb units,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0$$

(b)

For $r \to 0$, $\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R = 0 \Rightarrow R \propto r^l$.

For $r \to \infty$, $\frac{d^2R}{dr^2} + 2ER = 0 \Rightarrow R \propto \exp\left(-\sqrt{-2E}r\right)$.

(c) With $\ell = 0$, the preceding equation reduces to

$$\frac{d^2R}{dr^2} + 2ER + \frac{2}{r}\left(\frac{dR}{dr} + R\right) = 0$$

Substituting $R = C\exp(-r)$, we find $1 + 2E = 0$. From $\int_0^\infty R^2 r^2 dr = 1$ it follows that $C = 2$.