QM 15-Phys-710-001(/002/003) Spring 2000
Midterm: Spin. Magnetic Field.
Monday, May 1

1. For spin $s = 1/2$ find the raising and lowering operators $\widehat{s}_{\pm}$ and consider the result of their application to eigenfunctions ψ_{s_z}. Also, find $\widehat{s}_{\pm}^2$.

   Solution

$$\begin{aligned} \widehat{s}_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \widehat{s}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \widehat{s}_{\pm}^2 = 0 \\ \widehat{s}_+ \psi_{\frac{1}{2}} &= \widehat{s}_- \psi_{-\frac{1}{2}} = 0; \widehat{s}_- \psi_{\frac{1}{2}} = \psi_{-\frac{1}{2}}, \widehat{s}_+ \psi_{-\frac{1}{2}} = \psi_{\frac{1}{2}} \end{aligned}$$

2. For a system of two particles with spin $s = 1/2$, show that the operator

$$\widehat{P}_{10} = \frac{1 - 2\sigma_{1z}\sigma_{2z} + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{4}$$

   is a projection operator to the state $S = 1$, $S_z = 0$.

   Solution

$$\widehat{P}_{10} = \frac{1 - 2\sigma_{1z}\sigma_{2z} + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{4} = \frac{-\sigma_{1z}\sigma_{2z} + S(S+1) - 1}{2} = \begin{array}{l} \frac{1+1(1+1)-1}{2} = 1, \text{ when } S = 1, S_z = 0 \\ \frac{-1+1(1+1)-1}{2} = 0, \text{ when } S = 1, S_z = \pm 1 \\ \frac{1+0(0+1)-1}{2} = 0, \text{ when } S = 0, S_z = 0 \end{array}$$

3. Show that the Hamiltonian of a particle in a uniform magnetic field $\mathbf{H}$ can be written as

$$\widehat{H} = \frac{\widehat{\mathbf{p}}^2}{2m} - \frac{e\hbar}{2mc}\mathbf{H} \cdot \mathbf{l} + \frac{e^2}{8mc^2}(\mathbf{H} \times \mathbf{r})^2$$

   Hint: use the radial gauge.

   Solution

   Since $\nabla \cdot \mathbf{A} = \mathbf{0}$ for $\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{r}$, $\widehat{\mathbf{p}} \cdot \mathbf{A} - \mathbf{A} \cdot \widehat{\mathbf{p}} = -i\hbar\nabla \cdot \mathbf{A} = \mathbf{0}$ and

$$\begin{aligned} \widehat{H} &= \frac{\widehat{\mathbf{p}}^2}{2m} - \frac{e}{mc}\mathbf{A} \cdot \widehat{\mathbf{p}} + \frac{e^2}{2mc^2}\mathbf{A}^2 = \frac{\widehat{\mathbf{p}}^2}{2m} - \frac{e\hbar}{2mc}(\mathbf{H} \times \mathbf{r}) \cdot \widehat{\mathbf{p}} + \frac{e^2}{8mc^2}(\mathbf{H} \times \mathbf{r})^2 \\ &= \frac{\widehat{\mathbf{p}}^2}{2m} - \frac{e}{2mc}\mathbf{H} \cdot (\mathbf{r} \times \widehat{\mathbf{p}}) + \frac{e^2}{8mc^2}(\mathbf{H} \times \mathbf{r})^2 = \frac{\widehat{\mathbf{p}}^2}{2m} - \frac{e\hbar}{2mc}\mathbf{H} \cdot \mathbf{l} + \frac{e^2}{8mc^2}(\mathbf{H} \times \mathbf{r})^2 \end{aligned}$$