QM 15-Phys-710-001(/002/003) Winter 1999
Final
Monday, March 13

1. (20) A particle in the ground state of an infinitely deep potential well of width a $(0 < x < a)$ is subject to the perturbation of the following form:

$$V = V_0 \cos^2 \frac{\pi x}{a}$$

Find the shift of the energy in the lowest order perturbation theory.

Solution

The ground state wave function is

$$\psi^{(0)} = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$$

whereof

$$E^{(1)} = \frac{2V_0}{a} \int_0^a \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi x}{a} dx = \frac{V_0}{2a} \int_0^a \sin^2 \frac{2\pi x}{a} dx = \frac{V_0}{4}$$

2. (20) A particle is in the ground state of a shallow potential well of width a. Suddenly, the width of the well is changed to b (while the depth V_0 remains the same). What is the probability that the particle will remain in the well?

Hint: The wave function (and the energy) of the only bound state in a shallow well is approximately the same as the one in the potential $V = -\alpha\delta(x)$ with $\alpha = aV_0$; for the latter $\psi = \sqrt{\varkappa} \exp(-\varkappa |x|)$, where $\varkappa = \alpha m/\hbar^2$.

Solution

With the definitions $\varkappa_1 = aV_0 m/\hbar^2$, $\varkappa_2 = bV_0 m/\hbar^2$,

$$w = \varkappa_1 \varkappa_2 \left[\int_{-\infty}^{\infty} dx \exp\left(-(\varkappa_1 + \varkappa_2)|x|\right)\right]^2 = \frac{4\varkappa_1\varkappa_2}{(\varkappa_1 + \varkappa_2)^2} = \frac{4ab}{(a+b)^2}$$

3. (20) The equation for the radial part of the wave function in the attractive Coulomb potential is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + \frac{2m}{\hbar^2}\left(E + \frac{\alpha}{r}\right)R = 0$$

Find the normalized wave function $\psi(\mathbf{r})$ and the momentum distribution function of the ground state.

Hint: $dw(\mathbf{p}) = |\phi(\mathbf{p})|^2 d\mathbf{p}$, where $\phi(\mathbf{p}) = (2\pi\hbar)^{-3/2} \int d\mathbf{r} \psi(\mathbf{r}) \exp(-i\mathbf{p}\cdot\mathbf{r}/\hbar)$ is the wave function in the momentum representation.

Hint: Use atomic units.

Solution

In atomic units, setting $l = 0$, the equation reduces to

$$\frac{d^2R}{dr^2} + 2ER + \frac{2}{r}\left(\frac{dR}{dr} + R\right) = 0$$

Substituting $R = C\exp(-r)$, we find $1 + 2E = 0$. From $\int_0^\infty R^2 r^2 dr = 1$ it follows that $C = 2$ and

$$\psi(\mathbf{r}) = \sqrt{\frac{1}{4\pi}} 2\exp(-r) = \sqrt{\frac{1}{\pi}} \exp(-r)$$

Consequently, using $\ell = \hbar^2/m\alpha$,

$$\begin{aligned}
\phi(\mathbf{p}) &= \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{\pi}} \int d\mathbf{r} \exp(-r)\exp(-i\mathbf{p}\cdot\mathbf{r}) = \frac{2\pi}{(2\pi)^{3/2}}\sqrt{\frac{1}{\pi}} \int_0^\infty dr r^2 \exp(-r) \int_{-1}^{1} \exp(-ipr\cos\theta)\, d(\cos\theta) \\
&= \frac{i}{\sqrt{2}\pi p} \int_0^\infty dr r \exp(-r) \left[\exp(-ipr) - \exp(ipr)\right] = \frac{i}{\sqrt{2}\pi p}\left[\frac{1}{(1+ip)^2} - \frac{1}{(1-ip)^2}\right] \\
&= \frac{i}{\sqrt{2}\pi p}\left[\frac{-4ip}{(1+p^2)^2}\right] \to \frac{4}{\sqrt{2}\pi(1+p^2)^2}(\hbar/\ell)^{-3/2} \frac{4}{\sqrt{2}\pi\left(1 + \left(\frac{p}{(\hbar/\ell)}\right)^2\right)^2} = \frac{4}{\sqrt{2}\pi}(\hbar/\ell)^{5/2} \frac{1}{\left((\hbar/\ell)^2 + p^2\right)^2}
\end{aligned}$$