QM 15-Phys-710-001(/002/003) Spring 2000
Final: Tuesday, June 6

1. Find all possible terms and the normal term of the $(nd)^2$ electronic configuration (above the filled shell).

   Solution

   ${}^1S_0$, ${}^1D_2$, ${}^1G_4$, ${}^3P_{0,1,2}$, ${}^3F_{2,3,4}$; normal term ${}^3F_2$.

2. A particle with spin 1/2 moves in a constant magnetic field $\mathcal{H}$ (directed along the z axis). At the initial time ($t = 0$), the spin function has the form

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   At the time t:

   a) Find the spin function.
   b) Find the expectation value of the projection of the spin on the x and y axis.

   Note:

$$\widehat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \widehat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \widehat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Solution

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \mu_B \mathcal{H}\sigma_z \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \Rightarrow \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left[i\mu_B\mathcal{H}t/\hbar\right] \\ \exp\left[-i\mu_B\mathcal{H}t/\hbar\right] \end{pmatrix}$$

$$\overline{s}_x = \frac{1}{2}\begin{pmatrix} s_1^* & s_2^* \end{pmatrix}\sigma_x \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \frac{1}{2}\cos\left[2\mu_B\mathcal{H}t/\hbar\right], \; \overline{s}_y = -\frac{1}{2}\sin\left[2\mu_B\mathcal{H}t/\hbar\right]$$

3. Consider a system of two identical bosons with spin s.

   a) What are the possible values of the total spin S?
   b) What are the possible values of S for, respectively, even and odd values of the relative angular momentum is L?

   Hint: Interchange of particles results in the sign change $(-1)^L$ of the coordinate wave function.

   Solution

   $S = 2s,\ 2s-1,\ 2s-2,\ ...,\ 0$

$$(-1)^L = (-1)^S \Rightarrow \left\{ \begin{array}{l} S = 2s,\ 2s-2,\ \ldots\ ,\ 0, \text{ for } L \text{ even} \\ S = 2s-1,\ 2s-3,\ \ldots\ ,\ 1, \text{ for } L \text{ odd} \end{array} \right\}$$

4. Evaluate the commutator $[\widehat{v}_i, \widehat{v}_k]$ for a particle in the magnetic field.

   Solution

$$[\widehat{v}_i, \widehat{v}_k] = \frac{e}{m^2c}\left([\widehat{p}_k A_i] - [\widehat{p}_i A_k]\right) = \frac{ie\hbar}{m^2c}\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right) = \frac{ie\hbar}{m^2c}\epsilon_{ikl}\mathcal{H}_l$$

5. (Taking the eigenvalues of the diagonal operators) show that

$$\widehat{J}_i\widehat{L}_k\widehat{L}_i\widehat{J}_k = (\mathbf{J}\cdot\mathbf{L})^2 - \mathbf{J}\cdot\mathbf{L}$$

   Hint: Use $\left[\widehat{L}_i, \widehat{L}_k\right] = i\epsilon_{ikl}\widehat{L}_l$, $\left[\widehat{J}_i, \widehat{L}_k\right] = i\epsilon_{ikl}\widehat{L}_l$, $\left[\widehat{J}_i, \widehat{J}_k\right] = i\epsilon_{ikl}\widehat{J}_l$ and $\epsilon_{ikl}\epsilon_{ikm} = 2\delta_{lm}$.

   Solution

$$\begin{aligned}
\widehat{J}_i\widehat{L}_k\widehat{L}_i\widehat{J}_k &= \widehat{J}_i\widehat{L}_i\widehat{L}_k\widehat{J}_k + i\epsilon_{kil}\widehat{J}_i\widehat{L}_l\widehat{J}_k = (\mathbf{J}\cdot\mathbf{L})^2 + i\epsilon_{kil}\widehat{L}_l\widehat{J}_i\widehat{J}_k + (i\epsilon_{kil})(i\epsilon_{ilm})\widehat{L}_m\widehat{J}_k \\
&= (\mathbf{J}\cdot\mathbf{L})^2 + \frac{1}{2}i\epsilon_{kil}\widehat{L}_l\left(\widehat{J}_i\widehat{J}_k - \widehat{J}_k\widehat{J}_i\right) - \epsilon_{ilk}\epsilon_{ilm}\widehat{L}_m\widehat{J}_k = (\mathbf{J}\cdot\mathbf{L})^2 + \frac{1}{2}\left(i\epsilon_{kil}\widehat{L}_l\right)\left(i\epsilon_{ikm}\widehat{J}_m\right) - 2\delta_{km}\widehat{L}_m\widehat{J}_k \\
&= (\mathbf{J}\cdot\mathbf{L})^2 - \frac{1}{2}\epsilon_{kil}\epsilon_{ikm}\widehat{L}_l\widehat{J}_m - 2\mathbf{J}\cdot\ \mathbf{L} = (\mathbf{J}\cdot\mathbf{L})^2 + \frac{1}{2}\epsilon_{ikl}\epsilon_{ikm}\widehat{L}_l\widehat{J}_m - 2\mathbf{J}\cdot\ \mathbf{L} \\
&= (\mathbf{J}\cdot\mathbf{L})^2 + \delta_{lm}\widehat{L}_l\widehat{J}_m - 2\mathbf{J}\cdot\mathbf{L} = (\mathbf{J}\cdot\mathbf{L})^2 - \mathbf{J}\cdot\mathbf{L}
\end{aligned}$$