QM 15-Phys-710-001(/002/003) Fall 1999
Final
Wednesday, December 8

1. (25) In studying the emission of electrons from metals, it is necessary to take into account the fact that electrons with energy sufficient to escape from the metal can, according to quantum mechanics, undergo reflection at the surface of the metal. Consider a one-dimensional model with the potential

$$V = \begin{array}{l} -V_0, \text{ for } x < 0 \text{ (inside the metal)} \\ 0, \text{ for } x > 0 \text{ (outside the metal)} \end{array}$$

and determine the reflection coefficient of an electron of energy $E \geq 0$ at the surface of the metal. Investigate your answer in the following cases:

(a) $E = 0$; (b) $E \ll V_0$; (c) $E \gg V_0$;

and evaluate the reflection coefficient for $E = 0.1$ eV and $V_0 = 10$ eV.

Solution (see Goldman & Krivchenkov, §2 Problem 1 or Landau & Lifshitz, §25 Problem 1)

$$R = \left(\frac{\sqrt{E + V_0} - \sqrt{E}}{\sqrt{E + V_0} + \sqrt{E}} \right)^2 = \frac{V_0^2}{\left(\sqrt{E + V_0} + \sqrt{E} \right)^4}$$

The limiting cases

(a) $E = 0$:

$$R = 1$$

(b) $E \ll V_0$:

$$R \approx \left(\frac{\sqrt{V_0} - \sqrt{E}}{\sqrt{V_0} + \sqrt{E}} \right)^2 = \left(\frac{1 - \sqrt{\frac{E}{V_0}}}{1 + \sqrt{\frac{E}{V_0}}} \right)^2 \approx \left(1 - 2\sqrt{\frac{E}{V_0}} \right)^2 \approx 1 - 4\sqrt{\frac{E}{V_0}}$$

(c) $E \gg V_0$:

$$R \approx \frac{V_0^2}{\left(\sqrt{E} + \sqrt{E} \right)^4} = \frac{V_0^2}{16E^2}$$

Using (b), $R \approx 1 - 4\sqrt{.1/10} = .6$, for $E = 0.1$ eV and $V_0 = 10$ eV.

2. (20) Derive the wave functions and energies of the ground and first excited states of a one-dimensional harmonic oscillator.

Note: No answers, even correct ones, will be accepted without derivation.

Hint: One of the methods for solving the problem may be as follows:

- From the Schrodinger' equation, find the asymptotic behavior of the wave function at infinity;
- Look for a solution of the Schrodinger's equation as this asymptotic function times a prefactor;
- In thus obtained differential equation for the prefactor, look for a solution in the polynomial form. The latter should be of order zero (constant) for the ground state, of order one (linear function) for the first excited state, etc.

Solution

$$\psi'' + \left(\frac{2E}{\hbar\omega} - \xi^2 \right) \psi = 0, \; \xi = \sqrt{\frac{m\omega}{\hbar}} x \tag{0.1}$$

For large x,

$$\psi'' - \xi^2 \psi = 0$$

and

$$\psi \propto \exp\left(-\frac{\xi^2}{2}\right)$$

Substitute

$$\psi = \chi \exp\left(-\frac{\xi^2}{2}\right)$$

into eq. (0.1)

$$\chi^{''} - 2\xi\chi^{'} + 2n\chi = 0,\ n = \frac{E}{\hbar\omega} - \frac{1}{2}$$

and look for χ as power series.

(a) Ground state

$$\begin{aligned}\chi_0 &= a_0 \Rightarrow 2na_0 = 0 \Rightarrow n = 0 \Rightarrow E = \frac{1}{2}\hbar\omega \\ \psi &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)\end{aligned}$$

(b) First excited state

$$\begin{aligned}\chi_1 &= a_0 + a_1\xi \Rightarrow -2a_1\xi + 2n\left(a_0 + a_1\xi\right) = 0 \Rightarrow a_0 = 0 \text{ and } n = 1 \Rightarrow E = \frac{3}{2}\hbar\omega \\ \psi &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2^{1/2}\sqrt{\frac{m\omega}{\hbar}}\, x \exp\left(-\frac{m\omega x^2}{2\hbar}\right)\end{aligned}$$

3. (15 -) Prove the following relationships for a state whose z-component of the angular momentum is m:

(a) $\overline{l_x} = \overline{l_y} = 0$;

(b) $\overline{l_x l_y} = -\overline{l_y l_x} = im/2$;

(c) $\overline{\hat{l}_x^2} = \overline{\hat{l}_y^2}$.

Solution

See Assignment 3 Problem 3