QM 15-Phys-710-001(/002/003) Fall 1999
Quiz 2a: One-Dimensional Motion
Friday, November 12

1. Using Heisenberg's uncertainty relations, determine the lower limit of the possible values of the energy of a particle moving in a field of potential energy

$$V(x) = -\frac{V_0}{\cosh^2(\alpha x)}$$

Compare you result with the ground state energy given that the exact spectrum is as follows:

$$E_n = -\frac{\hbar^2\alpha^2}{8m}\left[-(1+2n) + \sqrt{1+\frac{8mV_0}{\hbar^2\alpha^2}}\right]^2$$

(Where n is limited by the condition that the expression inside the square brackets remains non-negative). You may assume that

$$V_0 \gg \frac{\hbar^2\alpha^2}{8m}$$

Bonus question: Find the exact energy spectrum.

Solution

$$\begin{aligned} E_0 &= -\frac{\hbar^2\alpha^2}{8m}\left[-1+\sqrt{1+\frac{8mV_0}{\hbar^2\alpha^2}}\right]^2 = -\frac{\hbar^2\alpha^2}{8m}\left(1+\frac{8mV_0}{\hbar^2\alpha^2}\right)\left[-\frac{1}{\sqrt{1+\frac{8mV_0}{\hbar^2\alpha^2}}}+1\right]^2 \\ &\simeq -\frac{\hbar^2\alpha^2}{8m}\left(\frac{8mV_0}{\hbar^2\alpha^2}\right)\left[-\frac{2}{\sqrt{\frac{8mV_0}{\hbar^2\alpha^2}}}+1\right] = -V_0 + 2\sqrt{\frac{\hbar^2\alpha^2}{8m}V_0} = -V_0 + \sqrt{\frac{\hbar^2\alpha^2}{2m}V_0} \end{aligned}$$

On the other hand,

$$V(x) = -\frac{V_0}{\cosh^2(\alpha x)} \simeq -\frac{V_0}{1+(\alpha x)^2} \simeq -V_0\left[1-(\alpha x)^2\right] = -V_0 + V_0(\alpha x)^2$$

And

$$\overline{E} = -V_0 + V_0\alpha^2\overline{x^2} + \frac{\overline{p^2}}{2m} \geq -V_0 + V_0\alpha^2\delta x^2 + \frac{\delta p^2}{2m} \geq -V_0 + V_0\alpha^2\delta x^2 + \frac{\hbar^2}{8m\delta x^2}$$

Minimizing with respect to δx^2,

$$\delta x^2_{\min} = \sqrt{\frac{\hbar^2}{8mV_0\alpha^2}}$$

So

$$\overline{E} \geq -V_0 + 2V_0\alpha^2\sqrt{\frac{\hbar^2}{8mV_0\alpha^2}} = -V_0 + \sqrt{\frac{\hbar^2\alpha^2}{2m}V_0}$$