

1. Using Heisenberg's uncertainty relations, determine the lower limit of the possible values of the energy of a particle moving in a field of potential energy

$$V(x) = -\frac{V_0}{\cosh^2(\alpha x)}$$

Compare your result with the ground state energy given that the exact spectrum is as follows:

$$E_n = -\frac{\hbar^2 \alpha^2}{8m} \left[ -(1+2n) + \sqrt{1 + \frac{8mV_0}{\hbar^2 \alpha^2}} \right]^2$$

(Where  $n$  is limited by the condition that the expression inside the square brackets remains non-negative). You may assume that

$$V_0 \gg \frac{\hbar^2 \alpha^2}{8m}$$

*Bonus question:* Find the exact energy spectrum.

*Solution*

$$\begin{aligned} E_0 &= -\frac{\hbar^2 \alpha^2}{8m} \left[ -1 + \sqrt{1 + \frac{8mV_0}{\hbar^2 \alpha^2}} \right]^2 = -\frac{\hbar^2 \alpha^2}{8m} \left( 1 + \frac{8mV_0}{\hbar^2 \alpha^2} \right) \left[ -\frac{1}{\sqrt{1 + \frac{8mV_0}{\hbar^2 \alpha^2}}} + 1 \right]^2 \\ &\simeq -\frac{\hbar^2 \alpha^2}{8m} \left( \frac{8mV_0}{\hbar^2 \alpha^2} \right) \left[ -\frac{2}{\sqrt{\frac{8mV_0}{\hbar^2 \alpha^2}}} + 1 \right] = -V_0 + 2\sqrt{\frac{\hbar^2 \alpha^2}{8m}} V_0 = -V_0 + \sqrt{\frac{\hbar^2 \alpha^2}{2m}} V_0 \end{aligned}$$

On the other hand,

$$V(x) = -\frac{V_0}{\cosh^2(\alpha x)} \simeq -\frac{V_0}{1 + (\alpha x)^2} \simeq -V_0 [1 - (\alpha x)^2] = -V_0 + V_0 (\alpha x)^2$$

And

$$\bar{E} = -V_0 + V_0 \alpha^2 \overline{x^2} + \frac{\overline{p^2}}{2m} \geq -V_0 + V_0 \alpha^2 \delta x^2 + \frac{\delta p^2}{2m} \geq -V_0 + V_0 \alpha^2 \delta x^2 + \frac{\hbar^2}{8m \delta x^2}$$

Minimizing with respect to  $\delta x^2$ ,

$$\delta x_{\min}^2 = \sqrt{\frac{\hbar^2}{8mV_0\alpha^2}}$$

So

$$\bar{E} \geq -V_0 + 2V_0 \alpha^2 \sqrt{\frac{\hbar^2}{8mV_0\alpha^2}} = -V_0 + \sqrt{\frac{\hbar^2 \alpha^2}{2m}} V_0$$