QM 15-Phys-710-001(/002/003) Fall 1999
Quiz 1: Operators in quantum mechanics
Wednesday, October 13

1. The projection operator $\widehat{P}(f_i)$ - an operator that projects to the state with a definite value f_i of the physical quantity f - is defined as follows:

$$\widehat{P}(f_i)\,\Psi_{f_k} = \delta_{f_i f_k}\Psi_{f_i} = \begin{matrix} \Psi_{f_i},\ f_i = f_k \\ 0,\ f_i \neq f_k \end{matrix}$$

where

$$\widehat{f}\Psi_{f_l} = f_l \Psi_{f_l}$$

Show that the operator $\widehat{P}(f_i)$ has the following properties:

a) $\widehat{P}(f_i)$ is Hermitian;

Hint: Consider

$$\int \Phi^* \widehat{P}(f_i)\,\Psi$$

where Φ and Ψ are two arbitrary wave functions and recall that any wave function can be expanded in terms of the eigenfunction of the operator $\widehat{f}$.

b) $\widehat{P}^2(f_i) = \widehat{P}(f_i)$.

Solution

a) Expanding Φ and Ψ,

$$\begin{aligned} \Psi &= \sum_n a_n \Psi_{f_n} \\ \Phi &= \sum_m b_m \Psi_{f_m} \end{aligned}$$

find

$$\begin{aligned} \int \Phi^* \widehat{P}(f_i)\,\Psi &= \int \sum_m b_m^* \Psi_{f_m}^* \widehat{P}(f_i) \sum_n a_n \Psi_{f_n} = \int \sum_m b_m^* \Psi_{f_m}^* \sum_n a_n \widehat{P}(f_i)\,\Psi_{f_n} \\ &= \int \sum_m b_m^* \Psi_{f_m}^* a_i \Psi_{f_i} = \sum_m a_i b_m^* \int \Psi_{f_m}^* \Psi_{f_i} = a_i b_i^* \end{aligned}$$

If the operator $\widehat{P}(f_i)$ is Hermitian, must have

$$\int \Phi^* \widehat{P}(f_i)\,\Psi = \int \left(\widehat{P}(f_i)\,\Phi\right)^* \Psi$$

but the latter can be written as

$$\begin{aligned} \int \left(\widehat{P}(f_i)\,\Phi\right)^* \Psi &= \int \left(\widehat{P}(f_i) \sum_m b_m \Psi_{f_m}\right)^* \sum_n a_n \Psi_{f_n} = \int (b_i \Psi_{f_i})^* \Psi_{f_n} \\ &= \sum_n b_i^* a_n \int \Psi_{f_i}^* \Psi_{f_n} = b_i^* a_i \end{aligned}$$

b)

$$\begin{aligned} \widehat{P}^2(f_i)\,\Psi &= \widehat{P}(f_i)\left(\widehat{P}(f_i)\,\Psi\right) = \widehat{P}(f_i)\left(\widehat{P}(f_i)\sum_n a_n \Psi_{f_n}\right) = \widehat{P}(f_i)\left(\sum_n a_n \widehat{P}(f_i)\,\Psi_{f_n}\right) \\ &= \widehat{P}(f_i)\,(a_i \Psi_{f_i}) = a_i \widehat{P}(f_i)\,\Psi_{f_i} = a_i \Psi_{f_i} = \widehat{P}(f_i)\,\Psi \end{aligned}$$