QM 15-030-710-003 Spring 1999
Midterm: Perturbation Theory.
Monday, May 3

1. (20 points) In the Hamiltonian of a harmonic oscillator,

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \frac{kx^2}{2} + \frac{\alpha x^2}{2}$$

consider $\alpha x^2/2$ formally as a perturbation.

a) Calculate the energy shift of the ground state and compare with the exact result.

b) Based on the exact result, evaluate the matrix element $\left(x^2\right)_{02}$.

Hint: Only $\left(x^2\right)_{00}$ and $\left(x^2\right)_{02}$ are non-zero.

c) (bonus) Since $\left(x^2\right)_{nm} = \sum_k x_{nk}x_{km}$ and only x_{nk} for $n = k+1$ and $n = k-1$ are non-zero, $\left(x^2\right)_{02} = x_{01}x_{12}$. Use b) and evaluate x_{01} and x_{12} explicitly to verify this result.

Solution

The exact energy of the ground state is

$$E_0 = \frac{\hbar}{2}\sqrt{\frac{k+\alpha}{m}} \approx \frac{\hbar\omega}{2}\left(1 + \frac{\alpha}{2k} - \frac{\alpha^2}{8k^2} + \ldots\right)$$

where the second equality is the expansion for $\alpha \ll k$. Treating $\alpha x^2/2$ as a perturbation, we find $\left(a^2 = \hbar/m\omega\right)$

$$\begin{aligned} E_0^{(1)} &= \frac{\alpha\left(x^2\right)_{00}}{2} = \frac{\alpha}{2}\int_{-\infty}^{\infty} x^2 \left(\sqrt{\pi}a\right)^{-1} \exp\left(-\left(x/a\right)^2\right) dx \\ &= \frac{\alpha a^2}{4} = \frac{\alpha\hbar}{4m\omega} = \frac{\hbar\omega}{2}\frac{\alpha}{2m\omega^2} = \frac{\hbar\omega}{2}\frac{\alpha}{2k} \end{aligned}$$

in agreement with the expansion of the exact result. Obviously,

$$E_0^{(2)} = -\frac{\hbar\omega}{2}\frac{\alpha^2}{8k^2} = -\left[\frac{\alpha}{2}\left(x^2\right)_{02}\right]^2 / 2\hbar\omega$$

so

$$\left(x^2\right)_{02} = \frac{\hbar\omega}{\sqrt{2}k}$$

On the other hand, direct evaluation gives

$$x_{01} = \frac{a}{\sqrt{2}},\ x_{12} = a$$

so

$$x_{01}x_{12} = \frac{a^2}{\sqrt{2}} = \frac{\hbar}{\sqrt{2}m\omega} = \frac{\hbar\omega}{\sqrt{2}k} = \left(x^2\right)_{02}$$

2. (20 points) A linear oscillator in the ground state begins, at $t = 0$, begins moving with a constant velocity v . Find the probability that the oscillator will remain in the ground state.

Hint: Consider the problem in the frame of reference moving with the oscillator and notice that, in this frame, the original WF can be written as $\widetilde{\psi}_0\left(x'\right) = \exp\left(-imvx'/\hbar\right)\psi_0\left(x'\right)$ where x' is the coordinate in the moving frame $\left(x' = x - vt\right)$ while the new WF of the ground state is $\psi_0\left(x'\right)$.

Solution

Using $a^2 = \hbar/m\omega$,

$$\begin{aligned} w &= \left|\int_{-\infty}^{\infty} \psi_0^*\left(x'\right)\widetilde{\psi}_0\left(x'\right)dx'\right|^2 = \left|\int_{-\infty}^{\infty} \psi_0^*\left(x'\right)\exp\left(-imvx'/\hbar\right)\psi_0\left(x'\right)dx'\right|^2 \\ &= \left|\int_{-\infty}^{\infty} \left(\sqrt{\pi}a\right)^{-1} \exp\left(-\left(x/a\right)^2\right)\exp\left(-imvx/\hbar\right)dx\right|^2 \\ &= \exp\left(-m^2a^2v^2/2\hbar^2\right) = \exp\left(-mv^2/2\hbar\omega\right) \end{aligned}$$