QM 15-030-710-002 Winter 1998
Final Exam
Monday, March 15

(15 + 15 points)

1. A particle with spin 1/2 moves in a time-varying uniform magnetic field directed along the z axis. The time variation of the field is given by some arbitrary function $\mathcal{H} = \mathcal{H}(t)$. At the initial time ($t = 0$), the spin function has the form

$$\begin{pmatrix} \exp(-i\alpha)\cos\delta \\ \exp(i\alpha)\sin\delta \end{pmatrix}$$

At the time t:

a) Find the spin function;

b) Find the mean value of the projection of the spin on the x and y axis;

c) Find the direction along which the projection has a definite value.

Solution

Denote spin function

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \mu_B \mathcal{H}\sigma_z \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

whereof

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \exp\left[-i\left(\alpha - \frac{\mu_B}{\hbar}\int_0^t dt\mathcal{H}(t)\right)\right]\cos\delta \\ \exp\left[i\left(\alpha - \frac{\mu_B}{\hbar}\int_0^t dt\mathcal{H}(t)\right)\right]\sin\delta \end{pmatrix}$$

Projection to axes

$$\begin{aligned} \bar{s}_x &= \frac{1}{2}\; s_1^* \quad s_2^* \;\sigma_x \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\ &= \frac{1}{2}\sin 2\delta\cos\left[2\left(\frac{\mu_B}{\hbar}\int_0^t dt\mathcal{H}(t) - \alpha\right)\right] \end{aligned}$$

$$\bar{s}_y = -\frac{1}{2}\sin 2\delta\sin\left[2\left(\frac{\mu_B}{\hbar}\int_0^t dt\mathcal{H}(t) - \alpha\right)\right]$$

Direction along which projection is 1/2 is

$$\theta = 2\delta,\ \phi = 2\left(\frac{\mu_B}{\hbar}\int_0^t dt\mathcal{H}(t) - \alpha\right)$$

The letter is found by solving equation (as done in class)

$$\widehat{s}_n\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

where

$$\begin{aligned} \widehat{s}_n &= \frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{n} \\ \boldsymbol{\sigma} &= \{\ \sigma_x,\quad \sigma_y,\quad \sigma_z\ \} \\ \mathbf{n} &= \{\ \sin\theta\cos\phi,\quad \sin\theta\sin\phi,\quad \cos\theta\ \} \end{aligned}$$

Note: The direction of projection 1/2 generates a conical surface and, in the case of a constant field, $\mathcal{H}(t) = \mathcal{H} = const$, it rotates with the constant frequency $2\mu_B\mathcal{H}/\hbar$ around the direction of the field.

2. For the Hamiltonian of a charged, spinless article in the magnetic field,

$$\widehat{H} = \frac{1}{2m}\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{r})$$

a) Find the expression for the velocity operator

$$\widehat{\mathbf{v}} = \frac{i}{\hbar}\left[\widehat{H}, \mathbf{r}\right]$$

b) Express the Hamiltonian in terms of velocity operators

c) Show that

$$[\widehat{v}_i, \widehat{v}_k] = \frac{ie\hbar}{m^2c}\epsilon_{ikl}\mathcal{H}_l$$

d) Find the energy of the particle in a uniform magnetic field using the known result for the harmonic oscillator.

Hint: Introduce new variables, $\widehat{v}_x = \alpha\widehat{Q}$, $\widehat{v}_y = \alpha\widehat{P}$, such that $\left[\widehat{P}, \widehat{Q}\right] = -i$, and express the Hamiltonian in terms of these variables.

Solution

$$\widehat{\mathbf{v}} = \frac{i}{\hbar}\left[\widehat{H}, \mathbf{r}\right] = \frac{1}{m}\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)$$

$$\widehat{H} = \frac{1}{2m}\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{r}) = \frac{m}{2}\widehat{\mathbf{v}}^2 + V(\mathbf{r})$$

$$[\widehat{v}_i, \widehat{v}_k] = \frac{e}{m^2c}\left([\widehat{p}_k A_i] - [\widehat{p}_i A_k]\right) = \frac{ie\hbar}{m^2c}\left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}\right) = \frac{ie\hbar}{m^2c}\epsilon_{ikl}\mathcal{H}_l$$

From

$$[\widehat{v}_y, \widehat{v}_x] = \alpha^2\left[\widehat{P}, \widehat{Q}\right] = -\frac{ie\hbar}{m^2c}\mathcal{H}$$

find

$$\alpha = \sqrt{\frac{e\hbar\mathcal{H}}{m^2c}}$$

and

$$\widehat{H} = \frac{e\hbar\mathcal{H}}{2mc}\left(\widehat{P}^2 + \widehat{Q}^2\right) + \frac{m}{2}\widehat{v}_z^2 \equiv \hbar\omega_{\mathcal{H}}\left(\frac{\widehat{P}^2}{2} + \frac{\widehat{Q}^2}{2}\right) + \frac{m}{2}\widehat{v}_z^2$$

with

$$\left[\widehat{P}, \widehat{Q}\right] = -i$$

Energy eigenvalues

$$E_n = \hbar\omega_{\mathcal{H}}\left(n + \frac{1}{2}\right) + \frac{m}{2}\widehat{v}_z^2 = \hbar\frac{e\mathcal{H}}{mc}\left(n + \frac{1}{2}\right) + \frac{m}{2}\widehat{v}_z^2$$