QM 15-030-710-001 Fall 1998
Final Exam
Wednesday, December 9

(11 + 11 + 11 + 12 points)

1. Consider a shallow

$$U_0 \ll \frac{\hbar^2}{ma^2}$$

symmetrical potential well

$$U(x) = \begin{matrix} -U_0, |x| < a \\ 0, \quad |x| > a \end{matrix}$$

Obtain the approximate expressions for the energy and the normalized wave function of the discrete spectrum level in this well and determine the probability of finding the particle inside the well.

SOLUTION

$$\psi(x) = \begin{matrix} A\cos\left(\sqrt{2m(U_0 - |E|)/\hbar^2}x\right), & |x| < a \\ B\exp\left(-\sqrt{2m|E|/\hbar^2}\,|x|\right), & |x| > a \end{matrix}$$

From continuity of WF and its derivative at $x = a$ (the above is an *even* function),

$$\sqrt{(U_0 - |E|)}\tan\left(\sqrt{2m(U_0 - |E|)a^2/\hbar^2}\right) = \sqrt{|E|}$$

For

$$|E|, U_0 \ll \frac{\hbar^2}{ma^2}$$

"tan" can be replaced by its argument,

$$\frac{\sqrt{|E|}}{\sqrt{(U_0 - |E|)}} \approx \sqrt{\frac{2m(U_0 - |E|)a^2}{\hbar^2}} \ll 1$$

which simplifies to

$$\frac{\sqrt{|E|}}{\sqrt{U_0}} \approx \sqrt{\frac{2mU_0a^2}{\hbar^2}} \ll 1$$

and

$$|E| \approx \frac{2mU_0a^2}{\hbar^2}U_0 \ll U_0$$

The approximate WF is

$$\begin{aligned} \psi(x) &\approx B\exp\left(-\sqrt{\frac{2m|E|}{\hbar^2}}\,|x|\right) \\ B &= \left(\frac{2m|E|}{\hbar^2}\right)^{1/4} = \left(\frac{2mU_0}{\hbar^2}a\right)^{1/2} \end{aligned}$$

and is virtually constant inside the well. The probability to find the particle inside the well is

$$\int_{-a}^{a}\psi^2(x)\,dx \approx 2B^2a = 2\frac{2mU_0a^2}{\hbar^2} \ll 1$$

The small probability of finding the particle inside the well is related to the slow decay of the WF in the region of the well so that the particle can be found with appreciable probability up to distances of order $\sqrt{\hbar^2/2m|E|} \gg a$ (in the classically inaccessible region).

2. For a potential profile

$$U(x) = \begin{matrix} U_0 > 0, x > 0 \\ 0, \quad x < 0 \end{matrix}$$

find the wave functions (up to a normalization constant) of the stationary states with energies $E < U_0$. Do these states form a complete set? What are the transmission and reflection coefficients for such states?

SOLUTION

$$\psi(x) = \begin{matrix} A\sin(kx+\delta), \; x < 0 \\ C\exp(-\varkappa x), \; x > 0 \end{matrix}$$

where

$$\begin{aligned} k &= \sqrt{\frac{2mE}{\hbar^2}} \\ \varkappa &= \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} \end{aligned}$$

From continuity of $\psi(x)$ and $\psi^{'}(x)$ at $x = 0$,

$$\begin{aligned} A\sin\delta &= C \\ kA\cos\delta &= -\varkappa C \end{aligned}$$

yielding

$$\tan\delta = -\frac{k}{\varkappa}$$

and (for $0 > \delta \geq -\pi/2$)

$$C = -A\frac{k}{\sqrt{k^2+\varkappa^2}}$$

3. Find the following commutator:

$$\left[\widehat{l}_i, a\widehat{\mathbf{r}} + b\widehat{\mathbf{p}}\right]$$

Specifically, consider the case $i = 1$ ($i = x$).

SOLUTION

$$\left[\widehat{l}_i, a\widehat{x}_k + b\widehat{p}_k\right] = i\varepsilon_{ikl}\left(a\widehat{x}_l + b\widehat{p}_l\right)$$

For $i = 1$, in coordinate representation,

$$\begin{aligned} \left[\widehat{l}_x, ay + b\widehat{p}_y\right] &= i\left(az + b\widehat{p}_z\right) \\ \left[\widehat{l}_x, az + b\widehat{p}_z\right] &= -i\left(ay + b\widehat{p}_y\right) \end{aligned}$$

4. In a state ψ_{lm} with definite values of the angular monument l and its projection m along the z-axis, find the mean values $\overline{\widehat{l}_x^2}$ and $\overline{\widehat{l}_y^2}$.

SOLUTION

$$\overline{\widehat{l}_x^2} = \overline{\widehat{l}_y^2} = \frac{\overline{\widehat{\mathbf{l}}^2} - \overline{\widehat{l}_z^2}}{2} = \frac{l(l+1) - m^2}{2}$$