QM 15-030-710-001 Winter 1998
Quiz 4: Spin.
Monday, March 1

(20 points)

1. Express $\left(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2\right)^2$ in a form containing no higher powers of Pauli matrices $\widehat{\boldsymbol{\sigma}}_{1,2}$ than one (here indices $1, 2$ denote that the corresponding matrices operate on spin variables of particles 1 and 2 respectively).

   Hint: Consider the operator

   $$\prod_i \left(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 - \lambda_i\right)$$

   where λ_i are the eigenvalues of the operator $\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2$.

   Solution

   $$\widehat{\mathbf{S}}^2 = \frac{1}{4}\left(\widehat{\boldsymbol{\sigma}}_1 + \widehat{\boldsymbol{\sigma}}_2\right)^2$$

   and, using $\widehat{\boldsymbol{\sigma}}^2 = 3$,

   $$\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 = -3 + 2\widehat{\mathbf{S}}^2$$

   so that the eigenvalues are

   $$\lambda_{1,2} = -3 + 2S\left(S+1\right) = \begin{array}{c} -3,\, S = 0 \\ 1,\, S = 1 \end{array}$$

   Consider now

   $$\left(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 - 1\right)\left(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 + 3\right) = 0$$

   Consequently,

   $$\left(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2\right)^2 = 3 - 2\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2$$