QM 15-030-710-001 Winter 1998
Quiz 4: Spin.
Monday, March 1

(20 points)

1. Express $(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2)^2$ in a form containing no higher powers of Pauli matrices $\widehat{\boldsymbol{\sigma}}_{1,2}$ than one (here indices $1, 2$ denote that the corresponding matrices operate on spin variables of particles 1 and 2 respectively).

   Hint: Consider the operator
$$\prod_i (\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 - \lambda_i)$$
   where λ_i are the eigenvalues of the operator $\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2$.

   Solution
$$\widehat{\mathbf{S}}^2 = \frac{1}{4} (\widehat{\boldsymbol{\sigma}}_1 + \widehat{\boldsymbol{\sigma}}_2)^2$$
   and, using $\widehat{\boldsymbol{\sigma}}^2 = 3$,
$$\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 = -3 + 2\widehat{\mathbf{S}}^2$$
   so that the eigenvalues are
$$\lambda_{1,2} = -3 + 2S(S+1) = \begin{matrix} -3, \; S = 0 \\ 1, \; S = 1 \end{matrix}$$
   Consider now
$$(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 - 1)(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2 + 3) = 0$$
   Consequently,
$$(\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2)^2 = 3 - 2\widehat{\boldsymbol{\sigma}}_1 \cdot \widehat{\boldsymbol{\sigma}}_2$$