QM 15-030-710-001 Winter 1998
Quiz 3: Angular Momentum. Central Field.
Monday, February 8

$(12 + 18$ points)

1. Consider two weakly interacting systems with quantum numbers $(l_1 = 1, m_1 = 1)$ and $(l_2 = 1, m_2 = 0)$.

   - Find the possible values of the total moment L of the combined system $(1 + 2)$ and evaluate $\overline{\widehat{\mathbf{L}}}$ and $\overline{\widehat{\mathbf{L}}^2}$;
   - Find the probabilities $w(L)$ of the possible values of L.

   Hint: $\overline{\widehat{\mathbf{L}}^2} = \sum_L L(L+1)\, w(L)$, $\sum_L w(L) = 1$.

   Solution

   - Possible values of L are 1 and 2. $\overline{L}_{x,y} = 0$, $\overline{L}_z = 1$. $\overline{\widehat{\mathbf{L}}^2} = l_1(l_1+1) + l_2(l_2+1) + 2m_1 m_2 = 4$.
   - $4 = 2w(2) + 6(1 - w(2)) = 6 - 4w(2)$ so that $w(2) = 1/2$ and $w(1) = 1 - w(2) = 1/2$.

2. For a spherical harmonic oscillator, $U = kr^2/2$,

   - Write the equation for the radial function (radial part of the wave function);
   - Find the asymptotic behavior of the radial function for $r \to 0$ and $r \to \infty$;
   - Find the energy and the wave function of the ground state.

   Solution

   - $$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}\left(E - \frac{kr^2}{2}\right)\right] R = 0$$
   - $r \to 0$, $R \propto r^\alpha$

     $$\begin{aligned} \left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right] R &= 0 \\ \alpha(\alpha-1) + 2\alpha - l(l+1) &= 0 \\ \alpha &= l \end{aligned}$$

     $r \to \infty$, $R \propto \exp\left(-r^2/a^2\right)$

     $$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2m}{\hbar^2}\left(-\frac{kr^2}{2}\right)\right] R = 0$$

     and collecting terms $\propto r^2$ in the brackets,

     $$\left(\frac{2}{a^2}\right)^2 = \frac{mk}{\hbar^2}$$
   - $l = 0$, $n_r = 0$, $R = C \exp - \left(r^2/a^2\right)$,

     $$\left[-\frac{2}{a^2} + \frac{4r^2}{a^4} - \frac{4}{a^2} + \frac{2m}{\hbar^2}\left(E - \frac{kr^2}{2}\right)\right] R$$

     and collecting the terms $\propto r^0$ (terms $\propto r^2$ give equation for a above),

     $$E = \frac{3\hbar^2}{ma^2} = \frac{3}{2}\frac{\hbar\sqrt{mk}}{m} = \frac{3}{2}\hbar\omega$$