QM 15-030-710-001 Fall 1998
Quiz 2: One-Dimensional Motion
Monday, November 9

(13 + 12 points)

1. Consider a shallow
$$U_0 \ll \frac{\hbar^2}{ma^2}$$
symmetrical potential well
$$U(x) = \begin{array}{l} -U_0,\ |x| < a \\ 0,\quad |x| > a \end{array}$$
Show that there is only one discrete spectrum level in this well and find the approximate expressions for its energy and the normalized wave function.

Solution

$$\psi(x) = \begin{array}{ll} A\cos\left(\sqrt{2m\left(U_0 - |E|\right)/\hbar^2}x\right), & |x| < a \\ B\exp\left(-\sqrt{2m\,|E|\,/\hbar^2}\,|x|\right), & |x| > a \end{array}$$

From continuity of WF and its derivative at $x = a$ (the above is an *even* function),
$$\sqrt{(U_0 - |E|)}\tan\left(\sqrt{2m\left(U_0 - |E|\right)a^2/\hbar^2}\right) = \sqrt{|E|}$$
For
$$|E|\,, U_0 \ll \frac{\hbar^2}{ma^2}$$
"tan" can be replaced by its argument,
$$\frac{\sqrt{|E|}}{\sqrt{(U_0 - |E|)}} \approx \sqrt{\frac{2m\left(U_0 - |E|\right)a^2}{\hbar^2}} \ll 1$$
which simplifies to
$$\frac{\sqrt{|E|}}{\sqrt{U_0}} \approx \sqrt{\frac{2mU_0a^2}{\hbar^2}} \ll 1$$
and
$$|E| \approx \frac{2mU_0a^2}{\hbar^2}U_0 \ll U_0$$

2. Using the variational method, find the approximate energy of the ground state of a harmonic oscillator using the trial functions of the following form:
$$\psi(x) = \frac{A}{\left(1 + \frac{x^2}{a^2}\right)}$$

Solution

$$\begin{aligned}
A^2 &= \frac{2}{\pi a}, \\
\overline{T} &= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\left(\psi'(x)\right)^2 = \frac{\hbar^2}{4ma^2} \\
\overline{U} &= \frac{m\omega^2}{2}\int_{-\infty}^{\infty} dx x^2\psi^2(x) = \frac{m\omega^2a^2}{2} \\
\overline{E}(a) &= \frac{\hbar^2}{4ma^2} + \frac{m\omega^2a^2}{2} \\
\frac{\partial\overline{E}(a_0)}{\partial a} &= 0,\ a_0^2 = \frac{1}{\sqrt{2}}\frac{\hbar}{m\omega} \\
\overline{E}(a_0) &= \min\left\{\overline{E}(a)\right\} = \frac{\hbar\omega}{\sqrt{2}}
\end{aligned}$$