**QM 15-030-710-001 Fall 1998**
**Quiz 1: Operators in quantum mechanics**
**Wednesday, October 11**

**Set 1**

1. Show that an arbitrary operator $\widehat{F}$ can be written as $\widehat{F} = \widehat{A} + i\widehat{B}$, where $\widehat{A}$ and $\widehat{B}$ are Hermitian operators.

   *Hint*: Consider $\widehat{F}^{\dagger}$, in addition to $\widehat{F}$.

$$\widehat{A} = \frac{\widehat{F} + \widehat{F}^{\dagger}}{2},\ \widehat{B} = \frac{\widehat{F} - \widehat{F}^{\dagger}}{2i}$$

2. Show that the mean values of the Hermitian operators $\widehat{L}^{\dagger}\widehat{L}$ and $\widehat{L}\widehat{L}^{\dagger}$, where $\widehat{L}$ is a linear operator, are not negative.

$$\overline{\widehat{L}^{\dagger}\widehat{L}} = \int \Psi^{*}\widehat{L}^{\dagger}\widehat{L}\Psi = \int \left(\widehat{L}^{*}\Psi^{*}\right)\widehat{L}\Psi = \int \left(\widehat{L}\Psi\right)^{*}\widehat{L}\Psi = \int \left|\widehat{L}\Psi\right|^{2}$$

$$\overline{\widehat{L}\widehat{L}^{\dagger}} = \int \Psi^{*}\widehat{L}\widehat{L}^{\dagger}\Psi = \int \left(\widetilde{\widehat{L}}\Psi^{*}\right)\widehat{L}^{\dagger}\Psi = \int \left(\widehat{L}^{\dagger}\Psi\right)^{*}\widehat{L}^{\dagger}\Psi = \int \left|\widehat{L}^{\dagger}\Psi\right|^{2}$$

**Set 2**

1. The commutator of the operators $\widehat{A}$ and $\widehat{B}$ of two physical quantities is given by $\left[\widehat{A},\widehat{B}\right] = i\widehat{C}$, where $\widehat{C}$ is Hermitian. Assuming, for simplicity that $\overline{A} = \overline{B} = 0$, show that

$$\overline{\widehat{A}^{2}}\ \overline{\widehat{B}^{2}} \geqslant \overline{\widehat{A}^{2}}\ \overline{\widehat{B}^{2}} \geqslant \frac{\overline{C}^{2}}{4}$$

   where all the mean values are with respect to the same, arbitrary state. Consider, in particular, the operators $\widehat{x}$ and $\widehat{p}$ and find the explicit form of the wave function for which $\overline{\widehat{x}^{2}}\ \overline{\widehat{p}^{2}}$ takes on its minimal value.

   *Hint*: Consider the integral $\int \left|\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right|^{2} dx$.

$$\begin{aligned}
\int \left|\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right|^{2} dx &= \int \left[\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right]\left[\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right]^{*} dx = \int \left[\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right]\left[\left(\alpha\widehat{A}^{*} + i\widehat{B}^{*}\right)\psi^{*}\right] dx \\
&= \int \psi^{*}\left(\alpha\widetilde{\widehat{A}^{*}} + i\widetilde{\widehat{B}^{*}}\right)\left(\alpha\widehat{A} - i\widehat{B}\right) dx = \int \psi^{*}\left[\left(\alpha\widehat{A}^{\dagger} + i\widehat{B}^{\dagger}\right)\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right] dx \\
&= \int \psi^{*}\left[\left(\alpha\widehat{A} + i\widehat{B}\right)\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right] dx = \alpha^{2}\int \psi^{*}\widehat{A}^{2}\psi - i\alpha\int \psi^{*}\left[\widehat{A},\widehat{B}\right]\psi + \int \psi^{*}\widehat{B}^{2}\psi dx \\
&= \alpha^{2}\overline{\widehat{A}^{2}} - i\alpha\overline{\left[\widehat{A},\widehat{B}\right]} + \overline{\widehat{B}^{2}} = \alpha^{2}\overline{\widehat{A}^{2}} + \alpha\overline{\widehat{C}} + \overline{\widehat{B}^{2}} \geqslant 0
\end{aligned}$$

   for any $\alpha$, hence,

$$\overline{\widehat{A}^{2}}\ \overline{\widehat{B}^{2}} \geqslant \frac{\overline{C}^{2}}{4}$$

   with "=" when

$$\left(\alpha\widehat{A} - i\widehat{B}\right)\psi = 0$$

   For $\widehat{x}$ and $\widehat{p}$

$$\overline{\widehat{x}^{2}}\ \overline{\widehat{p}^{2}} \geqslant \frac{\overline{\hbar}^{2}}{4}$$

   with "=" when

$$\left(\alpha x - \hbar\frac{d}{dx}\right)\psi = 0$$

   whereof ($\alpha < 0$)

$$\psi(x) = C\exp\left[\frac{x^{2}}{2\left(\hbar/\alpha\right)}\right] = C\exp\left[-\frac{x^{2}}{2\left(\hbar/\left|\alpha\right|\right)}\right]$$