QM 15-030-710-001 Fall 1998
Quiz 1: Operators in quantum mechanics
Wednesday, October 11

Set 1

1. Show that an arbitrary operator $\widehat{F}$ can be written as $\widehat{F} = \widehat{A} + i\widehat{B}$, where $\widehat{A}$ and $\widehat{B}$ are Hermitian operators.
   Hint: Consider $\widehat{F}^{\dagger}$, in addition to $\widehat{F}$.
$$\widehat{A} = \frac{\widehat{F} + \widehat{F}^{\dagger}}{2},\ \widehat{B} = \frac{\widehat{F} - \widehat{F}^{\dagger}}{2i}$$

2. Show that the mean values of the Hermitian operators $\widehat{L}^{\dagger}\widehat{L}$ and $\widehat{L}\widehat{L}^{\dagger}$, where $\widehat{L}$ is a linear operator, are not negative.
$$\overline{\widehat{L}^{\dagger}\widehat{L}} = \int \Psi^* \widehat{L}^{\dagger}\widehat{L}\Psi = \int \left(\widehat{L}^*\Psi^*\right)\widehat{L}\Psi = \int \left(\widehat{L}\Psi\right)^*\widehat{L}\Psi = \int \left|\widehat{L}\Psi\right|^2$$
$$\overline{\widehat{L}\widehat{L}^{\dagger}} = \int \Psi^* \widehat{L}\widehat{L}^{\dagger}\Psi = \int \left(\widetilde{\widehat{L}}\Psi^*\right)\widehat{L}^{\dagger}\Psi = \int \left(\widehat{L}^{\dagger}\Psi\right)^*\widehat{L}^{\dagger}\Psi = \int \left|\widehat{L}^{\dagger}\Psi\right|^2$$

Set 2

1. The commutator of the operators $\widehat{A}$ and $\widehat{B}$ of two physical quantities is given by $\left[\widehat{A}, \widehat{B}\right] = i\widehat{C}$, where $\widehat{C}$ is Hermitian. Assuming, for simplicity that $\overline{A} = \overline{B} = 0$, show that
$$\overline{\widehat{A}^2}\ \overline{\widehat{B}^2} \geqslant \overline{\widehat{A}^2}\ \overline{\widehat{B}^2} \geqslant \frac{\overline{C}^2}{4}$$
   where all the mean values are with respect to the same, arbitrary state. Consider, in particular, the operators $\widehat{x}$ and $\widehat{p}$ and find the explicit form of the wave function for which $\overline{\widehat{x}^2}\ \overline{\widehat{p}^2}$ takes on its minimal value.
   Hint: Consider the integral $\int \left|\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right|^2 dx$.

$$\begin{aligned}
\int \left|\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right|^2 dx &= \int \left[\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right]\left[\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right]^* dx = \int \left[\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right]\left[\left(\alpha\widehat{A}^* + i\widehat{B}^*\right)\psi^*\right] dx \\
&= \int \psi^* \left(\alpha\widetilde{\widehat{A}^*} + i\widetilde{\widehat{B}^*}\right)\left(\alpha\widehat{A} - i\widehat{B}\right) dx = \int \psi^* \left[\left(\alpha\widehat{A}^{\dagger} + i\widehat{B}^{\dagger}\right)\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right] dx \\
&= \int \psi^* \left[\left(\alpha\widehat{A} + i\widehat{B}\right)\left(\alpha\widehat{A} - i\widehat{B}\right)\psi\right] dx = \alpha^2 \int \psi^* \widehat{A}^2\psi - i\alpha \int \psi^* \left[\widehat{A}, \widehat{B}\right]\psi + \int \psi^* \widehat{B}^2\psi dx \\
&= \alpha^2\overline{\widehat{A}^2} - i\alpha\overline{\left[\widehat{A}, \widehat{B}\right]} + \overline{\widehat{B}^2} = \alpha^2\overline{\widehat{A}^2} + \alpha\overline{\widehat{C}} + \overline{\widehat{B}^2} \geqslant 0
\end{aligned}$$

   for any α, hence,
$$\overline{\widehat{A}^2}\ \overline{\widehat{B}^2} \geqslant \frac{\overline{C}^2}{4}$$
   with "=" when
$$\left(\alpha\widehat{A} - i\widehat{B}\right)\psi = 0$$
   For $\widehat{x}$ and $\widehat{p}$
$$\overline{\widehat{x}^2}\ \overline{\widehat{p}^2} \geqslant \frac{\hbar^2}{4}$$
   with "=" when
$$\left(\alpha x - \hbar\frac{d}{dx}\right)\psi = 0$$
   whereof ($\alpha < 0$)
$$\psi(x) = C\exp\left[\frac{x^2}{2(\hbar/\alpha)}\right] = C\exp\left[-\frac{x^2}{2(\hbar/|\alpha|)}\right]$$