Mechanics, Spring 2001
Final
Thursday, June 7

1. Two clocks located at the origin of the K and K' systems (which have a relative speed v) are synchronized when the origins coincide. After a time t, an observer at the origin of the K system observes the K' clock by means of the telescope. What does the K' clock read?

   Hint: it takes time for signal from clock K' to arrive to the origin of K.

   Solution

   The signal that arrives at the origin of K at t has started at time τ at the distance $v\tau$ from the origin and travels with the speed of light c. Therefore,

   $$t = \tau + \frac{v\tau}{c}$$

   and

   $$\tau = \frac{t}{1 + v/c}$$

   At the same time, the clock in K' is slower (moving frame) and it reads

   $$t' = \tau\sqrt{1-(v/c)^2} = t\frac{\sqrt{1-(v/c)^2}}{1+v/c} = t\sqrt{\frac{1-v/c}{1+v/c}}$$

   Notice that $t \to 2\tau$ as $v \to c$.

2. The relativistic momentum is given by

   $$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-(v/c)^2}}$$

   Evaluate the three-dimensional force vector

   $$\mathbf{f} = \frac{d\mathbf{p}}{dt}$$

   a) when the velocity changes only in direction (that is, suppose the force is perpendicular to the velocity);

   b) when the velocity changes only in magnitude (that is, the force is parallel to the velocity);

   Hint: The motion is one-dimensional - can choose all vectors along, say, x-axis.

   c) in general case, when the velocity changes both in magnitude and direction (check that you obtain a) and b) under proper assumptions).

   Hint: $dv^2/dt^2 = d\left(\mathbf{v}\cdot\mathbf{v}\right)/dt^2 = 2\mathbf{v}\cdot\left(d\mathbf{v}/dt\right)$.

   Solution

   a)

   $$\mathbf{f} = \frac{d\mathbf{p}}{dt} = \frac{m}{\sqrt{1-(v/c)^2}}\frac{d\mathbf{v}}{dt}$$

   b)

   $$\begin{aligned} f &= \frac{dp}{dt} = \frac{d}{dt}\frac{mv}{\sqrt{1-(v/c)^2}} = \frac{m}{\sqrt{1-(v/c)^2}}\frac{dv}{dt} + \frac{mv^2}{c^2\left(1-(v/c)^2\right)^{3/2}}\frac{dv}{dt} \\ &= \frac{m}{\sqrt{1-(v/c)^2}}\frac{dv}{dt}\left(1+\frac{(v/c)^2}{1-(v/c)^2}\right) = \frac{m}{\left(1-(v/c)^2\right)^{3/2}}\frac{dv}{dt} \end{aligned}$$

c)

$$\begin{aligned}\mathbf{f} &= \frac{d\mathbf{p}}{dt} = \frac{m}{\sqrt{1-(v/c)^2}}\frac{d\mathbf{v}}{dt} + \frac{m}{c^2\left(1-(v/c)^2\right)^{3/2}}\mathbf{v}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) \\ &= \frac{m}{\sqrt{1-(v/c)^2}}\left[\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}\left(\mathbf{v}\cdot(d\mathbf{v}/dt)\right)/c^2}{\left(1-(v/c)^2\right)}\right]\end{aligned}$$

Cases a) and b) follow from

$$\mathbf{v}\cdot(d\mathbf{v}/dt) = 0$$

and

$$\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}\left(\mathbf{v}\cdot(d\mathbf{v}/dt)\right)/c^2}{\left(1-(v/c)^2\right)} = \frac{dv}{dt}\left(1 + \frac{(v/c)^2}{1-(v/c)^2}\right) = \frac{dv}{dt}\frac{1}{1-(v/c)^2}$$

respectively.

3. Consider an infinitely long continuous string with tension τ. A mass M is attached to the string at $x = 0$. If a wave train with velocity $v = \omega/k = \sqrt{\tau/\rho}$ is incident from the left, show that the reflection and transmission occur at $x = 0$ and that the coefficients are given by $R = \sin^2\theta$ and $T = \cos^2\theta$ where $\tan\theta = M\omega^2/2k\tau$.

   Hint: the wave equation is

$$\frac{\partial^2\psi}{\partial x^2} + k^2\psi = 0$$

   and the boundary conditions at $x = 0$ are as follows:

$$\begin{aligned}\psi_2(0) &= \psi_1(0) \\ \frac{\partial\psi_2}{\partial x} &\mid_{x=0} - \frac{\partial\psi_1}{\partial x}\mid_{x=0} = -\frac{M\omega^2}{\tau}\psi(0)\end{aligned}$$

   Solution

$$\psi_1(x) = \exp(ikx) + B\exp(-ikx)$$

   is the incident and reflected waves and

$$\psi_2(x) = C\exp(ikx)$$

   is the outgoing wave. The reflection and transmission coefficients are $R = |B|^2$ and $T = |C|^2$ respectively. From the b.c.

$$\begin{aligned}1 + B &= C \\ 1 - B - C &= \frac{iM\omega^2}{\tau k}C\end{aligned}$$

   whereof

$$C = \frac{1}{1 + iM\omega^2/2\tau k}$$

   and

$$\begin{aligned}T &= |C|^2 = \frac{1}{1 + (M\omega^2/2\tau k)^2} \\ R &= 1 - T = \frac{\left(M\omega^2/2\tau k\right)^2}{1 + (M\omega^2/2\tau k)^2}\end{aligned}$$