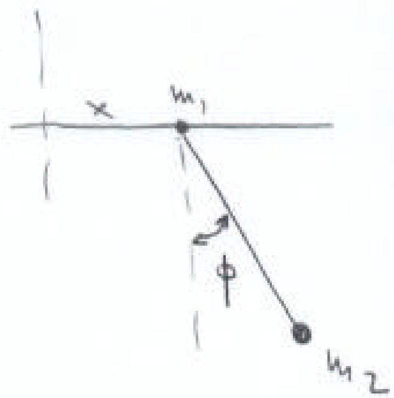

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi\right) + m_2 g l\cos\phi$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} = 0 \quad (x\text{ - ignorable variable})$$

$$(m_1+m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{const} \qquad (1)$$

Notice: c.m. $X = \dfrac{m_1 x + m_2(x + l\sin\phi)}{m_1+m_2}$

$\Rightarrow$ const $= (m_1+m_2)\bar{V}$, $\bar{V}$ velocity of c.m.

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\left[\frac{(m_1+m_2)\bar{V} - m_2 l\dot{\phi}\cos\phi}{(m_1+m_2)}\right]^2$$

$$+ \frac{1}{2}m_2\left[l^2\dot{\phi}^2 + 2l\,\frac{(m_1+m_2)\bar{V} - m_2 l\dot{\phi}\cos\phi}{(m_1+m_2)}\,\dot{\phi}\cos\phi\right]$$

$$+ m_2 g l\cos\phi$$

Making a small angle approx.

$$\mathcal{L} \simeq \frac{1}{2}(m_1+m_2)\left[\bar{V} - \frac{m_2}{m_1+m_2} l\dot{\phi}\right]^2$$

$$+\frac{1}{2} m_2 \left[l^2\dot{\phi}^2 + 2l\left(\bar{V} - \frac{m_2}{m_1+m_2} l\dot{\phi}\right)\dot{\phi}\right]$$

$$- \frac{m_2 g l \phi^2}{2}$$

omitting constant terms

$$= \cancel{-\bar{V} m_2 l\dot{\phi}} + \frac{1}{2}\frac{m_2^2}{m_1+m_2} l^2\dot{\phi}^2$$

$$+ \frac{1}{2}\frac{m_2(m_1-m_2)}{m_1+m_2} l^2\dot{\phi}^2 \cancel{+ m_2\bar{V} l\dot{\phi}} - \frac{m_2 g l \phi^2}{2}$$

$$= \frac{1}{2}\frac{m_1 m_2}{m_1+m_2} l^2\dot{\phi}^2 - \frac{m_2 g l \phi^2}{2}$$

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2} \Rightarrow \omega = \sqrt{\frac{g}{l}\frac{m_2}{\mu}}$$

Check: $m_1 \gg m_2 \Rightarrow \mu \simeq m_2$

and $\omega = \sqrt{\frac{g}{l}}$

Another method

Linearized EOM

$$\left.\begin{aligned} &(m_1+m_2)\ddot{x} + m_2 l \ddot{\phi} = 0 \\ &m_2 l^{\cancel{2}} \ddot{\phi} + m_2 \cancel{l} \ddot{x} + m_2 g \cancel{l} \phi = 0 \end{aligned}\right\}$$

$l\phi \equiv y$

$$\left.\begin{aligned} &-(m_1+m_2)\omega^2 x - m_2\omega^2 y = 0 \\ &-m_2\omega^2 x - m_2\omega^2 y + m_2 \frac{g}{l} y = 0 \end{aligned}\right\}$$

Characteristic equ.

$$\begin{vmatrix} -(m_1+m_2)\omega^2 & -m_2\omega^2 \\ -m_2\omega^2 & -m_2\omega^2 + m_2\frac{g}{l} \end{vmatrix} = 0$$

$$(m_1+m_2)\left(m_2\omega^2 - m_2\frac{g}{l}\right)\omega^2 - m_2^2\omega^4 = 0$$

a) $\omega^2 = 0 \Rightarrow$ c.m. motion with const speed

b) $m_1 m_2 \omega^2 - (m_1+m_2) m_2 \frac{g}{l} = 0$

$\omega = \sqrt{\frac{g}{l}}\sqrt{\frac{m_2}{\mu}}$, as before