LL vol. II §7 - Problem

Four-velocity

$$u^i = \left(\frac{1}{\sqrt{1-(v/c)^2}},\ \frac{\mathbf{v}}{c\sqrt{1-(v/c)^2}} \right)$$

Four-acceleration

$$\begin{aligned} w^i &= \frac{du^i}{ds} = \frac{1}{c\sqrt{1-(v/c)^2}} \frac{du^i}{dt} \\ &= \frac{1}{c\sqrt{1-(v/c)^2}} \left(\frac{\mathbf{v}\cdot(d\mathbf{v}/dt)}{c^2\left(1-(v/c)^2\right)^{3/2}},\ \frac{1}{c\sqrt{1-(v/c)^2}} \left[\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}\left(\mathbf{v}\cdot(d\mathbf{v}/dt)\right)/c^2}{\left(1-(v/c)^2\right)} \right] \right) \\ &= \frac{1}{c^2\left(1-(v/c)^2\right)^2} \left(\frac{\mathbf{v}}{c}\cdot\frac{d\mathbf{v}}{dt},\ \left[\frac{d\mathbf{v}}{dt}\left(1-\left(\frac{v}{c}\right)^2\right) + \frac{\mathbf{v}}{c^2}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) \right] \right) \end{aligned}$$

For $\mathbf{v} = v\widehat{\mathbf{x}}$,

$$\begin{aligned} w^i &= \frac{1}{c^2\left(1-(v/c)^2\right)^2} \left(\frac{v}{c}\frac{dv}{dt},\ \left[\frac{dv}{dt}\left(1-\left(\frac{v}{c}\right)^2\right) + \left(\frac{v}{c}\right)^2 \frac{dv}{dt} \right],\ 0,\ 0 \right) \\ &= \frac{1}{c^2\left(1-(v/c)^2\right)^2} \left(\frac{v}{c}\frac{dv}{dt},\ \frac{dv}{dt},\ 0,\ 0 \right) = \frac{dv/dt}{c^2\left(1-(v/c)^2\right)^2} \left(\frac{v}{c},\ 1,\ 0,\ 0 \right) \end{aligned}$$

Then

$$w^i w_i = -\frac{(dv/dt)^2}{c^4\left(1-(v/c)^2\right)^3} = -\frac{w^2}{c^4}$$

and

$$\frac{(dv/dt)}{\left(1-(v/c)^2\right)^{3/2}} = \frac{d}{dt}\frac{1}{\sqrt{1-(v/c)^2}} = w$$

where the second equality has been already derived above. The rest as discussed in class.