1° The derivatives of the function up to the second order should be continuous, the third piecewise continuous and besides,

$$\varphi(0) = \varphi(l) = 0\,; \quad \varphi''(0) = \varphi''(l) = 0\,. \tag{40}$$

For the convergence of the series

$$\sum_{n=1}^{\infty} n^k |\psi_n| \quad (k = -1, 0, 1)$$

the following requirements should be imposed on the initial velocity $\psi(x)$.

2° The function $\psi(x)$ is continuously differentiable, has a piecewise-continuous second derivative and besides

$$\psi(0) = \psi(l) = 0\,. \tag{41}$$

Thus we have proved that any vibration $u(x, t)$ with the initial functions $\varphi(x)$ and $\psi(x)$ satisfying the requirements 1° and 2° can be represented in the form of superposition of standing waves. The conditions 1° and 2° are sufficient conditions for the methods of proof adopted here.

A similar problem was solved in sub-section 5, § 2, by the method of propagating waves. The solution was

$$u(x, t) = \frac{\Phi(x - at) + \Phi(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\alpha)\, d\alpha\,, \tag{42}$$

where Φ and Ψ are extensions of the initial functions $\varphi(x)$ and $\psi(x)$ given in the segment $(0, l)$, which are odd with respect to 0 and l. The functions Φ and Ψ were shown to be periodic with a period $2l$ and therefore may be represented by the series

$$\Phi(x) = \sum_{n=1}^{\infty} \varphi_n \sin\frac{\pi n}{l} x, \quad \Psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\frac{\pi n}{l} x,$$

where φ_n and ψ_n are coefficients of the Fourier functions $\Phi(x)$ and $\Psi(x)$. Substituting these series in formula (42) and making use of the theorem of the sine and cosine of a sum and difference, we get

$$u(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos\frac{\pi n}{l} at + \frac{l}{\pi n a} \psi_n \sin\frac{\pi n}{l} at\right) \sin\frac{\pi n}{l} x, \tag{43}$$

which agrees with the expression obtained by the method of separation of variables.

Consequently formula (43) is valid with the same assumptions as formula (42) (see sub-section 1, § 3) which was obtained under the condition that the function $\Phi(x)$ is continuously differentiable twice and the function $\Psi(x)$ once.

For this to be true the functions $\varphi(x)$ and $\psi(x)$ must satisfy the following conditions

$$\varphi(0) = \varphi(l) = 0, \quad \psi(0) = \psi(l) = 0,$$
$$\varphi''(0) = \varphi''(l) = 0\,. \tag{44}$$

in addition to the conditions of differentiability.

Thus, conditions 1° and 2° which are sufficient for our proof of the method of separation of variables are more stringent than those sufficient to ensure the existence of a solution.

In justifying the possibility of representing the solution as a superposition of standing waves we used the method of proving the convergence of the series. The method can easily be applied to a number of other problems, although it imposes stricter restrictions than necessary on the initial functions.

4. *Inhomogeneous equations*

Let us consider the inhomogeneous wave equation

$$u_{tt} = a^2 u_{xx} + f(x, t), \quad a^2 = k/\varrho, \quad 0 < x < l \tag{45}$$

with the initial conditions

$$\left.\begin{aligned} u(x, 0) &= \varphi(x), \\ u_t(x, 0) &= \psi(x) \end{aligned}\right\} \; 0 \le x \le l \tag{46}$$

and the homogeneous boundary conditions

$$\left.\begin{aligned} u(0, t) &= 0, \\ u(l, t) &= 0, \end{aligned}\right\} \; t > 0\,. \tag{47}$$

We shall find a solution expressed as a Fourier series in x

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{\pi n}{l} x. \tag{48}$$

To find $u(x, t)$ it is necessary to determine the function $u_n(t)$. Let us represent the function $f(x, t)$ and the initial conditions as Fourier series:

$$\left.\begin{aligned} f(x,t) &= \sum_{n=1}^{\infty} f_n(t) \sin\frac{\pi n}{l} x, \quad & f_n(t) &= \frac{2}{l}\int_0^l f(\xi,t)\sin\frac{\pi n}{l}\xi\, d\xi; \\ \varphi(x) &= \sum_{n=1}^{\infty} \varphi_n \sin\frac{\pi n}{l} x, & \varphi_n &= \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{\pi n}{l}\xi\, d\xi; \\ \psi(x) &= \sum_{n=1}^{\infty} \psi_n \sin\frac{\pi n}{l} x, & \psi_n &= \frac{2}{l}\int_0^l \psi(\xi)\sin\frac{\pi n}{l}\xi\, d\xi. \end{aligned}\right\} \quad (49)$$

Substituting the trial solution (48) in equation (45)

$$\sum_{n=1}^{\infty} \sin\frac{\pi n}{l} x \left\{ -a^2 \left(\frac{\pi n}{l}\right)^2 u_n(t) + \ddot{u}_n(t) + f_n(t) \right\} = 0,$$

we see that it is satisfied if all the coefficients of the expansion are equal to zero, i. e.

$$\ddot{u}_n(t) + \left(\frac{\pi n}{l}\right)^2 a^2 u_n(t) = f_n(t). \quad (50)$$

We have obtained an ordinary differential equation with constant coefficients for $u_n(t)$. The initial conditions give

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} u_n(0)\sin\frac{\pi n}{l} x = \sum_{n=1}^{\infty} \varphi_n \sin\frac{\pi n}{l} x,$$

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty} \dot{u}_n(0)\sin\frac{\pi n}{l} x = \sum_{n=1}^{\infty} \psi_n \sin\frac{\pi n}{l} x$$

whence it follows that

$$\left.\begin{aligned} u_n(0) &= \varphi_n, \\ \dot{u}_n(0) &= \psi_n. \end{aligned}\right\} \quad (51)$$

These additional conditions completely determine the solution of the equation (50). The function $u_n(t)$ can be written as

$$u_n(t) = u_n^{(I)}(t) + u_n^{(II)}(t),$$

where

$$u_n^{(I)}(t) = \frac{l}{\pi n a}\int_0^t \sin\frac{\pi n}{l} a(t-\tau)\cdot f_n(\tau)\, d\tau \quad (52)$$

si the solution of the non-homogeneous equation with zero initial conditions* and

$$u_n^{(II)}(t) = \varphi_n \cos\frac{\pi n}{l} at + \frac{l}{\pi n a}\psi_n \sin\frac{\pi n}{l} at \quad (53)$$

is the solution of the homogeneous equation with the given initial conditions. Thus the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{l}{\pi n a}\int_0^t \sin\frac{\pi n}{l} a(t-\tau)\sin\frac{\pi n}{l} x\cdot f_n(\tau)\, d\tau +$$

$$+ \sum_{n=1}^{\infty}\left(\varphi_n \cos\frac{\pi n}{l} at + \frac{l}{\pi n a}\psi_n \sin\frac{\pi n}{l} at\right)\sin\frac{\pi n}{l} x. \quad (54)$$

The second sum represents a solution of the problem of free vibrations of a string with the given initial conditions and was investigated earlier. The first sum represents forced vibrations of a string under the action of an external force with zero initial conditions. Using expression (49) for $f_n(t)$ we find

$$u^{(I)}(x,t) = \int_0^t\int_0^l \left\{\frac{2}{l}\sum_{n=1}^{\infty}\frac{l}{\pi n a}\sin\frac{\pi n}{l} a(t-\tau)\sin\frac{\pi n}{l} x \sin\frac{\pi n}{l}\xi\right\} \times$$

$$\times f(\xi,\tau)\, d\xi\, d\tau = \int_0^t\int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau, \quad (55)$$

where

$$G(x,\xi,t-\tau) = \frac{2}{\pi a}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n}{l} a(t-\tau)\sin\frac{\pi n}{l} x \sin\frac{\pi n}{l}\xi. \quad (56)$$

We want to determine the physical significance of this solution. Let the function $f(\xi, \tau)$ differ from zero only in a small neighbourhood of the point $M_0(\xi_0, \tau_0)$:

$$\xi_0 \le \xi \le \xi_0 + \Delta\xi, \quad \tau_0 \le \tau \le \tau_0 + \Delta\tau.$$

* See paragraph in small type at the end of this section.

The function $\varrho f(\xi, \tau)$ represents the density of the force; the force applied to the region $(\xi_0, \xi_0 + \Delta\xi)$ is

$$F(\tau) = \varrho \int_{\xi_0}^{\xi_0+\Delta\xi} f(\xi, \tau)\, d\xi ,$$

and

$$I = \int_{\tau_0}^{\tau_0+\Delta\tau} F(\tau)\, d\tau = \varrho \int_{\tau_0}^{\tau_0+\Delta\tau} \int_{\xi_0}^{\xi_0+\Delta\xi} f(\xi, \tau)\, d\xi\, d\tau$$

is the impulse of this force during the time $\Delta\tau$. If the mean value theorem is applied to the expression

$$u(x, t) = \int_0^t \int_0^l G(x, \xi, t - \tau) f(\xi, \tau)\, d\xi\, d\tau =$$
$$= \int_{\tau_0}^{\tau_0+\Delta\tau} \int_{\xi_0}^{\xi_0+\Delta\xi} G(x, \xi, t - \tau) f(\xi, \tau)\, d\xi\, d\tau ,$$

we have

$$u(x, t) = G(x, \bar{\xi}, t - \bar{\tau}) \int_{\tau_0}^{\tau_0+\Delta\tau} \int_{\xi_0}^{\xi_0+\Delta\xi} f(\xi, \tau)\, d\xi\, d\tau , \tag{57}$$

where

$$\xi_0 < \bar{\xi} < \xi_0 + \Delta\xi, \quad \tau_0 < \bar{\tau} < \tau_0 + \Delta\tau .$$

In formula (57), going to the limit when $\Delta\xi \to 0$ and $\Delta\tau \to 0$, we get

$$u(x, t) = G(x, \xi_0, t - \tau_0) \frac{I}{\varrho} , \tag{58}$$

which can be considered as the effect of the instantaneous impulse of magnitude I.

If the function $(I/\varrho)\, G(x, \xi, t - \tau)$ represents the effect of a concentrated impulse, then it is immediately obvious that the effect of the continuously distributed force $f(x, t)$ must be represented by the formula

$$u(x, t) = \int_0^t \int_0^l G(x, \xi, t - \tau) f(\xi, \tau)\, d\xi\, d\tau , \tag{59}$$

which coincides with the formula (55).

The function giving the effect of a concentrated impulse for the infinite string was considered in the previous section. It is

a piecewise-constant function equal to $I/2a\varrho$ within the upper characteristic angle for the point (ξ, τ) and zero outside this angle. The function giving the effect of the concentrated impulse for the string with fixed ends $(0, l)$ may be obtained from the corresponding function for the infinite string by odd extension with respect to the points $x = 0$ and $x = l$.

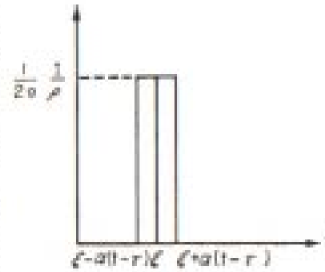

FIG. 23

Let us consider a moment t sufficiently close to τ, when the effect of reflections from the ends $x = 0$ and $x = l$ is still not felt. The impulse function is shown in the graph given in Fig. 23. We expand this function (putting $I = \varrho$) into a Fourier series in $\sin (\pi n/l)\, x$.

The coefficients of the Fourier series will be

$$A_n = \frac{2}{l} \int_0^l G(\alpha, \xi, t - \tau) \sin \frac{\pi n}{l} \alpha\, d\alpha = \frac{1}{al} \int_{\xi-a(t-\tau)}^{\xi+a(t-\tau)} \sin \frac{\pi n}{l} \alpha\, d\alpha =$$
$$= \frac{1}{a\pi n} \left\{ \cos \frac{\pi n}{l} [\xi - a(t - \tau)] - \cos \frac{\pi n}{l} [\xi + a(t - \tau)] \right\} =$$
$$= \frac{2}{a\pi n} \sin \frac{\pi n}{l} \xi \sin \frac{\pi n}{l} a (t - \tau) .$$

Hence we obtain the formula

$$G(x, \xi, t - \tau) = \frac{2}{\pi a} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{\pi n}{l} a (t - \tau) \sin \frac{\pi n}{l} x \cdot \sin \frac{\pi n}{l} \xi , \tag{60}$$

which coincides with formula (56) found by the method of separation of variables.

For the values $t \geqslant \tau$ when the effect of the fixed ends begins to be felt, it is cumbersome to construct the impulse function using characteristic curves; but the representation as a Fourier series is still valid.

We shall investigate the solution of the following problem without discussing the conditions of applicability of the formula obtained.

Let us consider a non-homogeneous linear equation with constant coeffcients

$$L(u) = u^{(n)} + p_1 u^{(n-1)} + \ldots + p_{n-1} u^{(1)} + p_n u = f(t) \qquad (1^*)$$

$$u^{(i)} = \left(\frac{d^i u}{dt^i}\right)$$

and the initial conditions

$$u^{(i)}(0) = 0 \quad (i = 0, 1, \ldots, n-1). \qquad (2^*)$$

Its solution is given by the formula

$$u(t) = \int_0^t U(t-\tau) f(\tau)\, d\tau, \qquad (3^*)$$

where $U(t)$ is the solution of a homogeneous equation

$$L(U) = 0$$

with the initial conditions

$$U^{(i)}(0) = 0 \ (i = 0, 1, \ldots, n-2), \ U^{(n-1)}(0) = 1. \qquad (4^*)$$

Calculating the derivatives of $u(t)$ by the differentiation of the right-hand side with respect to t, we find

$$\left.\begin{aligned}
u^{(1)}(t) &= \int_0^t U^{(1)}(t-\tau) f(\tau)\, d\tau + U(0) f(t) \quad [U(0) = 0],\\
u^{(2)}(t) &= \int_0^t U^{(2)}(t-\tau) f(\tau)\, d\tau + U^{(1)}(0) f(t) \quad [U^{(1)}(0) = 0],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
u^{(n-1)}(t) &= \int_0^t U^{(n-1)}(t-\tau) f(\tau)\, d\tau + U^{(n-2)}(0) f(t) \quad [U^{(n-2)}(0) = 0],\\
u^{(n)}(t) &= \int_0^t U^{(n)}(t-\tau) f(\tau)\, d\tau + U^{(n-1)}(0) f(t) \quad [U^{(n-1)}(0) = 1].
\end{aligned}\right\} \qquad (5^*)$$

Substituting these derivatives in equation (1*) we get:

$$L(u) = \int_0^t L[U(t-\tau)] f(\tau)\, d\tau + f(t) = f(t),$$

i. e. the equation is satisfied. It is obvious that the initial conditions (2*) are also fulfilled.

We can give a graphic physical interpretation of the function $U(t)$ and the formula (3*). Usually the function $u(t)$ represents the displacement of some system and $f(t)$ the force acting on this system. Let our system be in a state of rest for $t < 0$, and let the force be represented by

the function $f_\varepsilon(t)$ (> 0) which is different from zero only in the interval of time $0 < t < \varepsilon$. The impulse of this force will be denoted by

$$I = \int_0^\varepsilon f(\tau)\, d\tau.$$

Let $u_\varepsilon(t)$ be the function corresponding to $f_\varepsilon(t)$ considering ε as a parameter and putting $I = 1$. It can be shown that when $\varepsilon \to 0$ there is a $\lim\limits_{\varepsilon \to 0} u_\varepsilon(t)$ independent of the method of selecting $f_\varepsilon(t)$ and that this limit is equal to the function $U(t)$ defined above

$$U(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t),$$

if we put $U(t) = 0$ for $t < 0$. Thus the function $U(t)$ can naturally be called the function of the effect of an instantaneous impulse.

Considering the formula (3*) and applying the mean value theorem we get

$$u_\varepsilon(t) = U(t - \tau_\varepsilon^*) \int_0^\varepsilon f(\tau)\, d\tau = U(t - \tau_\varepsilon^*) \quad (0 \leqslant \tau_\varepsilon^* < \varepsilon < t).$$

Passing to the limit when $\varepsilon \to 0$, we see that there is a limit

$$\lim_{\varepsilon \to 0} u_\varepsilon(t) = \lim_{\varepsilon \to 0} U(t - \tau_\varepsilon^*) = U(t),$$

which proves our statement.

Now let us represent the solution of a non-homogeneous equation using $U(t)$ as the function of the effect of the instantaneous impulse. Dividing the interval $(0, t)$ by the points τ_l into equal parts

$$\Delta\tau = t/m,$$

we represent the function $f(t)$ in the form

$$f(t) = \sum_{l=1}^{m} f_l(t),$$

where

$$f_l(t) = \begin{cases} 0 & \text{where } t < \tau_l \text{ and } t \geqslant \tau_{l+1},\\ f(t) & \text{where } \tau_l \leqslant t < \tau_{l+1}. \end{cases}$$

Then the function

$$u(t) = \sum_{l=1}^{m} u_l(t),$$

where $u_l(t)$ are solutions of the equation $L(u_l) = f_l$ with zero initial data'

If m is sufficiently large then the function $u_l(t)$ may be considered as the function giving the effect of the instantaneous impulse with the intensity

$$I = f_l(\tau_l)\, \Delta\tau = f(\tau_l)\, \Delta\tau,$$

so that

$$u(t) = \sum_{i=1}^{m} U(t-\tau_i) f(\tau_i) \Delta\tau \underset{\Delta\tau\to 0}{\rightarrow} \int_0^t U(t-\tau) f(\tau) d\tau,$$

i. e. we arrive at the formula

$$u(t) = \int_0^t U(t-\tau) f(\tau) d\tau,$$

which shows that the effect of the continuously acting force may be represented by the superposition of the effects of instantaneous impulses.

In the case considered above $u_n^{(1)}$ satisfies the equation (50) and the conditions $u_n(0) = \dot{u}_n(0) = 0$. For the impulse function $U(t)$ we have:

$$\ddot{U} + \left(\frac{\pi n}{l}\right)^2 a^2 U = 0, \qquad U(0) = 0, \qquad \dot{U}(0) = 1,$$

so that

$$U(t) = \frac{l}{\pi n a} \sin \frac{\pi n}{l} at.$$

Hence (3*) we obtain the formula (52)

$$u_n^{(1)}(t) = \int_0^t U(t-\tau) f_n(\tau) = \frac{l}{\pi n a} \int_0^t \sin \frac{\pi n}{l} a(t-\tau) f_n(\tau) d\tau.$$

The integral representation (3*) of the solution of the ordinary differential equation (1*), obtained above, has the same physical significance as formula (59) which gives an integral representation of the solution of the homogeneous were equation of vibrations.

5. *First general boundary problem*

Let us consider the *first general boundary problem* for the wave equation:

$$u_{tt} = a^2 u_{xx} + f(x, t) \tag{45}$$

with the additional conditions

$$\left.\begin{aligned} u(x, 0) &= \varphi(x), \\ u_t(x, 0) &= \psi(x); \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} u(0, t) &= \mu_1(t), \\ u(l, t) &= \mu_2(t). \end{aligned}\right\} \tag{47'}$$

Let us introduce a new unknown function $v(x, t)$ putting

$$u(x, t) = U(x, t) + v(x, t),$$

so that $v(x, t)$ represents the deviation of the function $u(x, t)$ from some known function $U(x, t)$.

This function $v(x, t)$ will be defined as the solution of the equation

$$v_{tt} = a^2 v_{xx} + \bar{f}(x, t), \quad \bar{f}(x, t) = f(x, t) - [U_{tt} - a^2 U_{xx}]$$

with the additional conditions

$$\begin{aligned} v(x, 0) &= \bar{\varphi}(x), & \bar{\varphi}(x) &= \varphi(x) - U(x, 0), \\ v_t(x, 0) &= \bar{\psi}(x); & \bar{\psi}(x) &= \psi(x) - U_t(x, 0); \\ v(0, t) &= \bar{\mu}_1(t), & \bar{\mu}_1(t) &= \mu_1(t) - U(0, t), \\ v(l, t) &= \bar{\mu}_2(t); & \bar{\mu}_2(t) &= \mu_2(t) - U(l, t). \end{aligned}$$

Let us choose the auxiliary function $U(x, t)$ in such a way that

$$\bar{\mu}_1(t) = 0 \text{ or } \bar{\mu}_2(t) = 0;$$

a simple choice would be

$$U(x, t) = \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right].$$

The general boundary problem for the function $u(x, t)$ is thus reduced to a boundary problem for the function $v(x, t)$ with zero boundary conditions. The method of solution of this problem is described above (see sub-section 4).

6. *Boundary problems with stationary inhomogeneities*

Boundary problems with stationary inhomogeneities form a very important class of problems. Here the boundary conditions and the right-hand side of the equation do not depend on time

$$u_{tt} = a^2 u_{xx} + f_0(x), \tag{45'}$$

$$\left.\begin{aligned} u(x, 0) &= \varphi(x), \\ u_t(x, 0) &= \psi(x), \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} u(0, t) &= u_1, \\ u(l, t) &= u_2. \end{aligned}\right\} \tag{47'}$$

In this class it is natural to look for the solution as a sum

$$u(x, t) = \bar{u}(x) + v(x, t),$$