Green's function of wave operator

$$\square \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2$$

$$\square G = \delta(\vec{r}, t)$$

$$\left(\frac{\partial^2}{\partial t^2} + c^2 k^2\right) G(\vec{k}, t) = \frac{1}{(2\pi)^d} \delta(t)$$

$$G(\vec{k}, t) = \frac{\theta(t)}{(2\pi)^d} \frac{\sin ctk}{ck}$$

$$G(\vec{r}, t) = \frac{\theta(t)}{(2\pi)^d} \int d^d \vec{k} \, \frac{\sin ctk}{ck} e^{i\vec{k}\cdot\vec{r}}$$

1) $d = 3$

$$G(\vec{r}, t) = \frac{\theta(t)}{(2\pi)^3} (2\pi) \int_0^\infty dk \, k^2 \frac{\sin ctk}{ck} \int_0^\pi d\theta \sin\theta \, e^{ikr\cos\theta}$$

$$= \frac{\theta(t)}{(2\pi)^2} \int_0^\infty dk \, k^2 \frac{\sin ctk}{ck} \int_{-1}^{1} dz \, e^{ikrz}$$

$$= \frac{\theta(t)}{(2\pi)^2} \frac{1}{cr} \frac{1}{i} \int_0^\infty dk \, \underbrace{\sin ctk \overbrace{\left(e^{ikr} - e^{-ikr}\right)}^{\propto \sin kr}}_{\text{even function of } k \text{ on } [-\infty, \infty]}$$

$$= \frac{\theta(t)}{(2\pi)^2} \frac{1}{cr} \frac{1}{2i} \int_{-\infty}^{\infty} dk \, \sin ctk \left(e^{ikr} - e^{-ikr}\right)$$

$$= -\frac{\theta(t)}{(2\pi)^2}\frac{1}{cr}\frac{1}{4}\int_{-\infty}^{\infty} dk\,\left(e^{ictk}-e^{-ictk}\right)\left(e^{ikr}-e^{-ikr}\right)$$

because of $\theta(t)$

$$= -\frac{\theta(t)}{(2\pi)^2}\frac{1}{cr}\frac{(2\pi)2}{4}\left[\cancel{\delta(r+ct)} - \delta(r-ct)\right]$$

$$= \frac{\theta(t)}{2\pi}\frac{1}{c}\frac{1}{2r}\delta(ct-r) = \frac{\theta(t)}{4\pi c^2 t}\overbrace{\delta(ct-r)}^{\delta_{S_{ct}}}$$

$$= \frac{\theta(t)}{2\pi c}\delta(c^2t^2-r^2)^{*}$$

* $\delta(c^2t^2-r^2)\theta(t) = \frac{1}{2r}\delta(ct-r)\theta(t)$

2) $d = 1$

$$G(x,t) = \frac{\theta(t)}{2\pi} \int_{-\infty}^{\infty} dk \, \frac{\sin ctk}{ck} e^{ikx}$$

$$I = \int_{-\infty}^{\infty} dk \, \frac{\sin ctk}{k} e^{ikx}$$

$$\frac{\partial I}{\partial x} = i \int_{-\infty}^{\infty} dx \, \frac{e^{ictk} - e^{-ictk}}{2i} e^{ikx}$$

$$= \frac{2\pi}{2}\left[\delta(ct+x) - \delta(ct-x)\right]$$

$$I = \pi\left[\theta(ct+x) + \theta(ct-x)\right] + f(t) \quad (*)$$

$$\frac{\partial I}{\partial t} = c \int_{-\infty}^{\infty} dk \, \frac{e^{ictk} + e^{-ictk}}{2} e^{ikx}$$

$$= c\pi\left[\delta(ct+x) + \delta(ct-x)\right]$$

Compare with $(*)$ $\Rightarrow f'(t) = 0 \Rightarrow f(t) = C$

$$G(x,t) = \frac{\theta(t)}{2\pi c}\left\{\pi\left[\theta(ct+x) + \theta(ct-x)\right] + C\right\}$$

$$= \frac{\theta(ct - |x|)}{2c} + \frac{C\theta(t)}{2\pi c}$$

$$G(0,t) = \frac{\theta(t)}{2c} + \frac{C\theta(t)}{2\pi c} = \frac{\theta(t)}{2\pi}\int_{-\infty}^{\infty} dk \, \frac{\sin ctk}{ck} = \frac{\theta(t)}{2c} \Rightarrow C = 0$$