$$I_1 = \frac{\partial I}{\partial z} = \int_0^1 \left(\frac{t}{1-t}\right)^\lambda \frac{dt}{(t-z)^2}$$

$$I_1 \left(1 - e^{2\pi i \lambda}\right) = 2\pi i \,\mathrm{Res}\left[\left(\frac{t}{1-t}\right)^\lambda \frac{1}{(t-z)^2},\ t = z\right]$$

$$= 2\pi i \left\{ \lambda \left(\frac{t}{1-t}\right)^{\lambda-1} \left[\frac{1}{1-t} + \frac{t}{(1-t)^2}\right]\right\}\Bigg|_{t=z}$$

$$= 2\pi i \lambda \left(\frac{z}{1-z}\right)^{\lambda-1} \frac{1}{(1-z)^2}$$

$$I_1 e^{\pi i \lambda}(-2i \sin \pi \lambda) = 2\pi i \lambda e^{\pi i(\lambda+1)} \frac{z^{\lambda-1}}{(z-1)^{\lambda+1}}$$

$$I_1 = \frac{\pi\lambda}{\sin\pi\lambda} \frac{z^{\lambda-1}}{(z-1)^{\lambda+1}} = \frac{\partial I}{\partial z}$$

$$I = \left[\lambda \int \frac{dz\, z^{\lambda-1}}{(z-1)^{\lambda+1}} + \text{const}\right] \frac{\pi}{\sin\pi\lambda}$$

$$= \frac{\pi}{\sin\pi\lambda}\left[\lambda \int \frac{dz}{z^2}\left(\frac{z}{z-1}\right)^{\lambda+1} + \text{const}\right]$$

$x = \frac{1}{z}$

$$= \frac{\pi}{\sin\pi\lambda}\left[-\lambda \int dx \left(\frac{1}{1-x}\right)^{\lambda+1} + \text{const}\right]$$

$$= \frac{\pi}{\sin\pi\lambda}\left[-(1-x)^{-\lambda} + \text{const}\right]$$

$$= \frac{\pi}{\sin\pi\lambda}\left[-\left(1-\frac{1}{z}\right)^{-\lambda} + \text{const}\right] = \frac{\pi}{\sin\pi\lambda}\left[\text{const} - \frac{z^\lambda}{(z-1)^\lambda}\right]$$

const = 1 from $\lambda = 0$