

An eigen-value problem

$$Lu \equiv -(pu')' + qu = \lambda u, \quad 0 < x < l \quad (1)$$

$$\left. \begin{aligned} h_1 u(0) - h_2 u'(0) &= 0 \\ H_1 u(l) + H_2 u'(l) &= 0 \end{aligned} \right\} \quad (2)$$

is called the Sturm-Liouville problem

Here  $p(x) > 0, q(x) \geq 0$   
 $h_{1,2}, H_{1,2} \geq 0$   
 $h_1 + h_2, H_1 + H_2 > 0$

Green's function

Consider a b.v. problem,

$$Lu = f(x) \quad (3)$$

Let  $v_1, v_2$  be solutions of

$$\left. \begin{aligned} Lv &= 0 \\ h_1 v_1(0) - h_2 v_1'(0) &= 0 \\ H_1 v_2(l) + H_2 v_2'(l) &= 0 \end{aligned} \right\} \quad (4)$$

Assume that  $\lambda = 0$  is not an e.v., then  $v_1$  and  $v_2$  are linearly independent and

$$w(x) = \begin{vmatrix} v_1(x) & v_2(x) \\ v_1'(x) & v_2'(x) \end{vmatrix} \neq 0^*$$

\* Wronskian

(1)

Suppose  $k_i$  exist that

$$\sum_{i=1}^n k_i v_i = 0$$

Differentiate  $n$  times

$$\sum_{i=1}^n k_i v_i = 0$$

$$\sum_{i=1}^n k_i v_i' = 0$$

$$\sum_{i=1}^n k_i v_i^{(n-1)} = 0$$

$\Rightarrow W =$

$$\begin{vmatrix} v_1 & \dots & v_n \\ v_1' & \dots & v_n' \\ \vdots & & \vdots \\ v_1^{(n-1)} & \dots & v_n^{(n-1)} \end{vmatrix} = 0$$

(2)

The combination  $p(x)w(x)$  is a constant

$$\frac{d}{dx} [p(x)w(x)] = p'w + pw'$$

$$w' = (v_1 v_2' - v_1' v_2)' = v_1 v_2'' - v_1'' v_2$$

$$-p'v_1 - pv_1'' + qv_1 = 0, \quad v_1'' = -\frac{p'}{p}v_1' + \frac{q}{p}v_1$$

$$w' = v_1 \left( -\frac{p'}{p}v_2' + \frac{q}{p}v_2 \right) - v_2 \left( -\frac{p'}{p}v_1' + \frac{q}{p}v_1 \right)$$

$$\frac{d}{dx} [p(x)w(x)] = p'(v_1 v_2' - v_1' v_2)$$

$$+ v_1(-p'v_2' + qv_2) - v_2(-p'v_1' + qv_1) = 0$$

Use variational technique

$$u(x) = C_1(x)v_1(x) + C_2(x)v_2(x)$$

$$u'(x) = \underbrace{C_1'(x)v_1(x) + C_2'(x)v_2(x)}_0 + C_1(x)v_1'(x) + C_2(x)v_2'(x)$$

$$u''(x) = C_1'v_1' + C_2'v_2' + C_1v_1'' + C_2v_2''$$

$$-pu'' - p'u' + qu = f$$

$$C_1'v_1' + C_2'v_2' + C_1v_1'' + C_2v_2'' - \frac{p'}{p}C_1v_1' - \frac{p'}{p}C_2v_2' + \frac{q}{p}C_1v_1 + \frac{q}{p}C_2v_2 = \frac{f}{p}$$

Recall that  $v'' - \frac{p'}{p} v' + \frac{q}{p} v = 0$

(3)

Then,

$$\left. \begin{aligned} c_1' v_1 + c_2' v_2 &= -\frac{f}{p} \\ c_1' v_1 + c_2' v_2 &= 0 \end{aligned} \right\} \text{ w.r. to } c_1', c_2'$$

$$c_{1,2}' = \pm \frac{f(x) v_{2,1}(x)}{p(x) w(x)} = \pm \frac{f(x) v_{2,1}(x)}{p(0) w(0)}$$

B.v.

$$h_1 u(0) - h_2 u'(0) = \left\{ h_1 [c_1 v_1 + c_2 v_2] - h_2 [c_1 v_1' + c_2 v_2'] \right\} \Big|_0$$

Since  $c_1' v_1 + c_2' v_2 = 0$ ,

$$\begin{aligned} h_1 u(0) - h_2 u'(0) &= \underbrace{c_1 [h_1 v_1 - h_2 v_1']}_0 + c_2 [h_1 v_2 - h_2 v_2'] \Big|_0 \\ &= \left\{ c_2 [h_1 v_2 - h_2 v_2'] \right\} \Big|_0 \end{aligned}$$

Hence,  $c_2(0) = 0$ . Also, by analogy,  $c_1(0) = 0$

$$c_1(x) = -\frac{1}{p(x)w(x)} \int_x^l f(\gamma) v_2(\gamma) d\gamma$$

$$c_2(x) = -\frac{1}{p(x)w(x)} \int_0^x f(\gamma) v_1(\gamma) d\gamma$$

(4)

$$u(x) = \int_0^l G(x,y) f(y) dy$$

$$G(x,y) = -\frac{1}{p(x)w(x)} \begin{cases} v_1(x)v_2(y), & 0 \leq x \leq y \\ v_2(x)v_1(y), & y \leq x \leq l \end{cases}$$

SL problem is equiv. to e.v. problem for the integral equation

$$u(x) = \lambda \int_0^l G(x,y) u(y) dy$$