Midterm, MathPhys, 04/25/95

Problem 1

Solve the problem of the heating,

$$\nabla^2 T = \tfrac{1}{\kappa}\frac{\partial T}{\partial t},$$

of an infinite cilinder $0 \le r \le a$, with zero initial temperature, if a temperature T_0 is maintained at the surface. Find an approximate expression for the temperature, averaged over a cross-section of the cilinder, in the steady state (steady state = long times).

Problem 2

Find the transverse vibrations,

$$\nabla^2 u = c^2 \frac{\partial^2 u}{\partial t^2},$$

of a circular membrane $0 \le r \le a$ fixed around the edge, assuming that the initial deflection has the form of the paraboloid of convolution,

$$u(r,0) = A\left(1 - \tfrac{r^2}{a^2}\right),$$

and the initial velocities, $\frac{\partial u}{\partial t}(r,0)$, are zero.

Problem 3

Find the temperature distribution inside a rectangular thin lamina $0 \le x \le a$, $0 \le y \le b$, if a constant heat flux Q is fed into one of its sides, while the remaining three sides are maintained at a constant temperature T_0. The coefficient of thermal conductivity is K.

In the first two problems, take the integrals exactly. You may need the following relationships:

A. $\int x^n J_{n-1}(x)dx = x^n J_n(x)$; B. $J_{n\mp1} = \tfrac{n}{x}J_n \pm J_n'$; C. $c_n = \dfrac{\int_0^a f(y) J_n(x_{mn}y)\,y\,dy}{\frac{a^2}{2}\left[J_{n\pm1}(x_{mn}a)\right]^2}$

$T = T_0$ is a solution, $T = T_0 + R(\rho)J(t) \leftarrow J \propto e^{-\lambda t}$

$$\nabla^2 R + \frac{\lambda}{K} R = 0 \quad , \quad R \propto J_0\left(\sqrt{\frac{\lambda}{K}}\rho\right)$$

$$T(a,t) = T_0 \Rightarrow \sqrt{\frac{\lambda_n}{K}}\,a = x_{0n} : \text{n'th zero of } J_0$$

$$T(\rho,t) = T_0 + \sum_n A_n J_0\left(x_{0n}\frac{\rho}{a}\right) e^{-\frac{x_{0n}^2 K}{a^2}t}$$

$$T(\rho,0) = 0 = T_0 + \sum A_n J_0\left(x_{0n}\frac{\rho}{a}\right)$$

$$A_n = -T_0 \frac{\int_0^a J_0\left(x_n\frac{\rho}{a}\right)\rho\,d\rho}{\frac{a^2}{2}J_1^2(x_{0n})} = -T_0 \frac{x_{0n}\left(\frac{a}{x_{0n}}\right)^2 J_1(x_{0n})}{\frac{a^2}{2}J_1^2(x_{0n})} = -\frac{2T_0}{x_{0n}J_1(x_{0n})}$$

$$T = T_0\left(1 - 2\sum_n e^{-\frac{x_{0n}^2 K}{a^2}t} \frac{J_0\left(x_{0n}\frac{\rho}{a}\right)}{x_{0n}J_1(x_{0n})}\right)$$

$$\xrightarrow[\text{state}]{\text{steady}} T_0\left(1 - 2e^{-\frac{x_{01}^2 K}{a^2}t} \frac{J_0\left(x_{01}\frac{\rho}{a}\right)}{x_{01}J_1(x_{01})}\right)$$

$$\overline{T}(t) = \frac{\int_0^a T(\rho,t)\,2\pi\rho\,d\rho}{\pi a^2} = T_0\left(1 - 2e^{-\frac{x_{01}^2 K}{a^2}t} \frac{\int_0^a J_0\left(x_{01}\frac{\rho}{a}\right)2\pi\rho\,d\rho}{\pi a^2 x_{01}J_1(x_{01})}\right)$$

$$= T_0\left(1 - \frac{4}{x_{01}^2} e^{-\frac{x_{01}^2 K}{a^2}t}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right\}$$

$$u(r,0) = A\left(1-\frac{r^2}{r_0^2}\right), \quad \partial_t u(r,0) = 0, \quad u(a,t) = 0$$

$$u \propto R(r)T(t) \leftarrow T(t) \propto \cos\omega t, \quad \omega = ck$$

$$R'' + \frac{1}{r}R' + k^2 R = 0, \quad R \propto J_0(kr)$$

$$u(a,t) = 0 \Rightarrow ka = x_{0n}$$

$$u(r,t) = \sum_n B_n J_0\left(x_{0n}\frac{r}{a}\right)\cos\frac{c x_{0n}}{a}t$$

$$u(r,0) = A\left(1-\frac{r^2}{a^2}\right) = \sum_n B_n J_0\left(x_{0n}\frac{r}{a}\right)$$

$$B_n = A \frac{\int_0^a J_0\left(x_{0n}\frac{r}{a}\right)\left(1-\frac{r^2}{a^2}\right) r\,dr}{\frac{a^2}{2} J_1^2(x_{0n})}$$

$$= 2A \frac{\int_0^{x_{0n}} J_0(y)\left(1-\frac{y^2}{x_{0n}^2}\right) y\,dy}{x_{0n}^2 J_1^2(x_{0n})^2} = \frac{8A}{x_{0n}^3 J_1(x_{0n})}$$

$$\left\{\begin{aligned} &\int_0^{x_{0n}} J_0(y)\,y\,dy = J_1(y)y\Big|_0^{x_{0n}} = J_1(x_{0n})x_{0n} \\ &\int_0^{x_{0n}} J_0(y)y^3\,dy = \int_0^{x_{0n}} \left(\frac{J_1(y)}{y} + J_1'(y)\right)y^3\,dy \\ &= J_1(y)y^3\Big|_0^{x_{0n}} - 2\int_0^{x_{0n}} J_1(y)y^2\,dy = J_1(x_{0n})x_{0n}^3 + 2\int_0^{x_{0n}} J_0'(y)y^2\,dy \end{aligned}\right.$$

$$= J_1(x_{0n})x_{0n}^3 - 4J_1(x_{0n})x_{0n}$$

$$u(r,t) = 8A\sum_n \frac{J_0\left(x_{0n}\frac{r}{a}\right)\cos\frac{c x_{0n}}{a}t}{x_{0n}^3 J_1(x_{0n})}$$

$$\nabla^2 T = 0 \qquad T = T_0 + \mathcal{T}(x,y)$$

$$\nabla^2 \mathcal{T} = 0, \qquad \mathcal{T} = 0 \text{ for } \begin{cases} x = 0 \\ x = a \\ y = 0 \end{cases}$$

$$-K \frac{\partial \mathcal{T}}{\partial y}(x,b) = Q$$

$$\mathcal{T} = X(x)\,Y(y)$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \qquad \frac{X''}{X} = -\lambda^2$$

$$X = A \sin \lambda x + B \sin \lambda (a-x)$$

$$X(0) = 0 \Rightarrow B = 0$$

$$X(a) = 0 \Rightarrow \lambda a = \pi n$$

$$Y = C \sinh \lambda y$$

$$\mathcal{T} = \sum_n a_n \sin \lambda_n x \sinh \lambda_n y$$

$$-K \frac{\partial \mathcal{T}}{\partial y}(x,b) = Q = -K \sum a_n \lambda_n \cosh \lambda_n b \sin \lambda_n x$$

$$a_n \lambda_n \cosh \lambda_n b = -\frac{Q}{K} \frac{4}{\pi} \frac{1}{n}, \text{ if } n \text{ odd}$$

$$\mathcal{T} = -\frac{4Q}{K\pi^2} \sum_{m=0}^{\infty} \frac{\sin \frac{\pi(2m+1)}{a} x \sinh \frac{\pi(2m+1)}{a} y}{\cosh \frac{\pi(2m+1)}{a} b\,(2m+1)^2}$$