

A. $\xi(t) = \theta(t) z(t)$, where z satisfies

$$Lz = z^{(m)} + a_1(t)z^{(m-1)} + \dots + a_m(t)z = 0,$$

and

$$z(0) = z'(0) = \dots = z^{(m-2)}(0) = 0, \quad z^{(m-1)}(0) = 1,$$

satisfies in turn

$$L\xi = \delta(t).$$

Proof:

$$\begin{aligned} \xi' &= \theta(t)z'(t) + \delta(t)z(t) = \theta(t)z'(t) + z(0)\delta \\ &= \theta(t)z'(t) \end{aligned}$$

$$\xi^{(m-1)} = \theta(t)z^{(m-1)}(t)$$

$$\xi^{(m)} = \theta(t)z^{(m)}(t) + \delta(t)$$

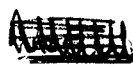
$$L\xi = \theta(t)Lz + \delta(t) = \delta(t)$$

Problems

$$L = \frac{d^2}{dx^2} - a^2$$

$$L = \left(\frac{d}{dx} + a\right)^m$$

$$L = \frac{d^2}{dx^2} \rightarrow \frac{d}{dx} + 1$$



$$\xi = \theta(x) \frac{\sinh ax}{a}$$

$$\xi = \theta(x) e^{\pm ax} \frac{x^{m-1}}{(m-1)!}$$

$$\xi = \theta(x) x e^x$$

B. Example: $L = \frac{d}{dt} + a$

1) $(\frac{d}{dt} + a)z = 0, z(0) = 1$

$$z = e^{-at}, \quad \varepsilon = \theta(t)e^{-at}$$

2) $L\varepsilon = \delta(t), \quad \mathcal{L}(L\varepsilon) = 1$

$$(s+a)\varepsilon(s) = 1, \quad \varepsilon(s) = \frac{1}{s+a} \quad (*)$$

$$\varepsilon(t) = \frac{1}{2\pi i} \int_C ds \frac{e^{st}}{s+a} = e^{-at} \theta(t) \quad (**)$$


3) $\varepsilon = \int d\omega e^{i\omega t} \varepsilon(\omega)$

$$(i\omega + a)\varepsilon(\omega) = \frac{1}{2\pi i}, \quad \varepsilon(\omega) = \frac{1}{2\pi i} \frac{1}{\omega - ia}$$

$$\varepsilon(t) = \int d\omega \frac{e^{i\omega t}}{2\pi i (\omega - ia)}$$

$$= \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases} = e^{-at} \theta(t)$$

(*) Integrate from $t = 0^-$, where $\varepsilon = 0$

(**) C:  to the right of $-a$

C. Example: $L = \frac{d^2}{dt^2} + a^2$

$$1) \left(\frac{d^2}{dt^2} + a^2\right) z = 0, \quad z(0) = 0, \quad z'(0) = 1$$

$$z = \frac{\sin at}{a}, \quad \varepsilon = \frac{\sin at}{a} \Theta(t)$$

$$2) L\varepsilon = \delta(t), \quad \mathcal{L}(L\varepsilon) = 1$$

$$(s^2 + a^2)\varepsilon(s) = 1, \quad \varepsilon(s) = \frac{1}{s^2 + a^2}$$

$$\varepsilon(t) = \frac{1}{2\pi i} \int_C ds \frac{e^{st}}{s^2 + a^2}$$

$$= \frac{\Theta(t)}{2\pi i} \left[2\pi i \frac{e^{iat}}{2ia} + 2\pi i \frac{e^{-iat}}{-2ia} \right]$$

$$= \Theta(t) \frac{\sin at}{a}$$

$$3) \varepsilon(\omega) = -\frac{1}{2\pi} \frac{1}{\omega^2 - a^2}$$

$$\varepsilon(t) = -\frac{1}{2\pi} \int d\omega \frac{e^{i\omega t}}{\omega^2 - a^2}$$

Move both poles above real axis

$$\varepsilon(t) = -\frac{2\pi i}{2\pi} \left[\frac{e^{iat}}{2a} + \frac{e^{-iat}}{-2a} \right] \Theta(t)$$

$$= \frac{\sin at}{a} \Theta(t)$$