A. $\mathcal{E}(t) = \theta(t)\, Z(t)$, where Z satisfies

$$LZ = Z^{(m)} + a_1(t) Z^{(m-1)} + \dots + a_m(t) Z = 0,$$

and

$$Z(0) = Z'(0) = \dots = Z^{(m-2)}(0) = 0, \quad Z^{(m-1)}(0) = 1,$$

satisfies in turn

$$L\mathcal{E} = \delta(t).$$

Proof:

$$\mathcal{E}' = \theta(t) Z'(t) + \delta(t) Z(t) = \theta(t) Z'(t) + Z(0)\delta$$

$$= \theta(t) Z'(t)$$

$$\vdots$$

$$\mathcal{E}^{(m-1)} = \theta(t) Z^{(m-1)}(t)$$

$$\mathcal{E}^{(m)} = \theta(t) Z^{(m)}(t) + \delta(t)$$

$$L\mathcal{E} = \theta(t) LZ + \delta(t) = \delta(t)$$

Problems

$$L = \frac{d^2}{dx^2} - a^2 \qquad \mathcal{E} = \theta(x) \frac{\sinh ax}{a}$$

$$L = \left(\frac{d}{dx} \mp a\right)^m \qquad \mathcal{E} = \theta(x) e^{\pm ax} \frac{x^{m-1}}{(m-1)!}$$

$$L = \frac{d^2}{dx^2} - 2\frac{d}{dx} + 1 \qquad \mathcal{E} = \theta(x) x e^x$$

B. Example: $L = \frac{d}{dt} + a$

1) $\left(\frac{d}{dt} + a\right) z = 0$, $z(0) = 1$

$z = e^{-at}$, $\mathcal{E} = \theta(t) e^{-at}$

2) $L\mathcal{E} = \delta(t)$, $\mathcal{L}(L\mathcal{E}) = 1$

$(s+a)\,\mathcal{E}(s) = 1$, $\mathcal{E}(s) = \frac{1}{s+a}$ (*)

$$\mathcal{E}(t) = \frac{1}{2\pi i}\int_C ds\, \frac{e^{st}}{s+a} = e^{-at}\theta(t) \quad (**)$$

3) $\mathcal{E} = \int d\omega\, e^{i\omega t}\, \mathcal{E}(\omega)$

$(i\omega + a)\,\mathcal{E}(\omega) = \frac{1}{2\pi}$, $\mathcal{E}(\omega) = \frac{1}{2\pi i}\,\frac{1}{\omega - ia}$

$$\mathcal{E}(t) = \int d\omega\, \frac{e^{i\omega t}}{2\pi i(\omega - ia)}$$

$$= \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases} = e^{-at}\theta(t)$$

(*) Integrate from $t = 0^-$, where $\mathcal{E} = 0$

(**) C:  to the right of $-a$

C. Example: $L = \frac{d^2}{dt^2} + a^2$

1) $\left(\frac{d^2}{dt^2} + a^2\right) z = 0$, $z(0) = 0$, $z'(0) = 1$

$$z = \frac{\sin at}{a}, \quad \mathcal{E} = \frac{\sin at}{a}\,\Theta(t)$$

2) $L\mathcal{E} = \delta(t)$, $\mathcal{L}(L\mathcal{E}) = 1$

$$(s^2 + a^2)\,\mathcal{E}(s) = 1, \quad \mathcal{E}(s) = \frac{1}{s^2 + a^2}$$

$$\mathcal{E}(t) = \frac{1}{2\pi i} \int_C ds \, \frac{e^{st}}{s^2 + a^2}$$

$$= \frac{\Theta(t)}{2\pi i}\left[2\pi i \, \frac{e^{iat}}{2ia} + 2\pi i \, \frac{e^{-iat}}{-2ia}\right]$$

$$= \Theta(t)\,\frac{\sin at}{a}$$

3) $\mathcal{E}(\omega) = -\frac{1}{2\pi}\,\frac{1}{\omega^2 - a^2}$

$$\mathcal{E}(t) = -\frac{1}{2\pi}\int d\omega \, \frac{e^{i\omega t}}{\omega^2 - a^2}$$

Move both poles above real axis

$$\mathcal{E}(t) = -\frac{2\pi i}{2\pi}\left[\frac{e^{iat}}{2a} + \frac{e^{-iat}}{-2a}\right]\Theta(t)$$

$$= \frac{\sin at}{a}\,\Theta(t)$$