Math Physics Final
Winter Quarter 1997

Problem 1

Find the retarded Green's function of the Schrödinger's operator

$$i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla^2$$

in a d-dimensional space.

Problem 2

Find the temperature of a rod $0 \le x \le l$ with thermally insulated lateral surface if the initial temperature is everywhere zero, if the temperature at the ends is held at zero, and if a source of constant strength Q is concentrated at the point x_0 (where $0 < x_0 < l$) in the rod.

$$\frac{\partial T}{\partial t}-\frac{K}{C\rho}\frac{\partial^2 T}{\partial x^2}=\frac{Q}{C\rho}\delta\left(x-x_0\right) \qquad (0 < x < l,\ t > 0)$$

$$T\left(0,t\right)=T\left(l,t\right)=0 \qquad (t > 0)$$

$$T\left(x,0\right)=0 \qquad (0 < x < l)$$

Use notation $\frac{K}{C\rho}=\kappa$.

Solutions

Problem 1

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\right)G(\vec r,t)=\delta(\vec r,t)$$

$$G(\vec r,t)=\int d\vec k\; G(\vec k,t)e^{i\vec k\cdot\vec r}$$

$$\delta(\vec r,t)=\frac{1}{(2\pi)^d}\int d\vec k\; e^{i\vec k\cdot\vec r}\,\delta(t)$$

$$i\hbar\frac{\partial G(\vec k,t)}{\partial t}-\frac{\hbar^2k^2}{2m}G(\vec k,t)=\frac{1}{(2\pi)^d}\delta(t)$$

$$\left(\frac{\partial}{\partial t}+\frac{i\hbar k^2}{2m}\right)G(\vec k,t)=-\frac{i}{\hbar(2\pi)^d}\delta(t)$$

$$G(\vec k,t)=-\frac{\hbar\, i\,\Theta(t)}{(2\pi)^d}\,e^{-i\frac{\hbar k^2}{2m}t}$$

$$G(\vec r,t)=-\frac{i\Theta(t)}{\hbar(2\pi)^d}\int d\vec k\; e^{i\vec k\cdot\vec r}e^{-i\frac{\hbar k^2}{2m}}$$

$$=-\frac{i\Theta(t)}{\hbar(2\pi)^d}\prod_i\int dk_i\; e^{ik_ix_i-i\frac{\hbar k_i^2}{2m}t}$$

$$=-\frac{i\Theta(t)}{\hbar(2\pi)^d}\,e^{i\frac{mr^2}{2\hbar t}}\prod_i\int dk_i\, e^{-\frac{i\hbar t}{2m}\left(k-\frac{mx_i}{\hbar t}\right)^2}$$

$$=-\frac{i\Theta(t)}{\hbar(2\pi)^d}\left(\frac{2\pi m}{i\hbar t}\right)^{d/2}e^{imr^2/2\hbar t}$$

Problem 1 (continued)

$$G(\vec{r},t) = -\frac{i\theta(t)}{\hbar}\left(\frac{m}{2\pi\hbar t}\right)^{d/2} e^{i\left(\frac{mr^2}{2\hbar t} - \frac{\pi d}{4}\right)}$$

Note: $m \to m + i0$ for convergence

Problem 2

$$\delta(x-x_0) = \frac{2}{\ell}\sum_n \sin\frac{n\pi x}{\ell}\sin\frac{n\pi x_0}{\ell}$$

$$T = \sum_n A_n(t)\sin\frac{n\pi x}{\ell}\sin\frac{n\pi x_0}{\ell}$$

$$\frac{\partial A_n}{\partial t} + \kappa\left(\frac{n\pi}{\ell}\right)^2 A_n = \frac{2Q}{C\rho\ell}, \qquad A_n(0) = 0$$

a) homogeneous eqn.

$$\frac{\partial A_n}{\partial t} + \kappa\left(\frac{n\pi}{\ell}\right)^2 A_n = 0, \quad A_n = A e^{-\kappa\left(\frac{n\pi}{\ell}\right)^2 t}$$

b) particular solution

$$A^*_n = \frac{2Q}{C\rho\ell}\left(\frac{\ell}{n\pi}\right)^2\frac{1}{\kappa} \qquad \kappa = \frac{K}{C\rho}$$

$$A_n = A e^{-\kappa\left(\frac{n\pi}{\ell}\right)^2 t} + \frac{2Q}{C\rho\ell}\left(\frac{\ell}{n\pi}\right)^2\frac{1}{\kappa}$$ and use $A_n(0)=0$

$$= \frac{2Q}{K\ell}\left(\frac{\ell}{n\pi}\right)^2\left(1 - e^{-\kappa\left(\frac{n\pi}{\ell}\right)^2 t}\right)$$