Math Physics
Final Exam
Spring 1996

Problem 1 **50 pts**

Find the Green's function of the Laplace equation

$$\nabla^2 u = 0$$

inside the sphere $r = a$, with the boundary condition $u(r = a) = 0$. Note that $G(\vec{r}, \vec{r}')$ can depend only on r, r', and θ, the angle between $\vec{r}$ and $\vec{r}'$.

Hint: For the radial equation, use the technique we worked out in class for the Green's function of the Sturm-Liouville operator.

Some useful formulae:

(a) $$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

(b) $$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]P_l(\cos\theta) = -l(l+1)P_l(\cos\theta)$$

(c) $$\delta(\Omega - \Omega') = \sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\theta)$$

To verify your result, use the expansion

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

to convert the series into a closed form expression and observe that it is equivalent to the formula obtained by the method of images.

Problem 2 **50 pts**

The temperature inside a sphere of radius a satisfies the conditions:

$$\nabla^2 T = \alpha^{-1}\frac{\partial T}{\partial t}$$

$$T = 0 \text{ at } t = 0$$

$$T = T_0 \text{ at } r = a$$

Find $T(r, \theta, t)$ for $t > 0$. Approximate your expression to be useful for long times.

Hint: You may use the expression for the Laplacian in spherical coordinates given in Problem 1.

Problem 3 **30 pts**

Find the spectral representation for the Green's function of the equation

$$(\nabla^2 - \lambda)u = 0$$

inside a square of size a on a side, with the boundary conditions

$$u(0, y) = u(a, y) = 0 \text{ and } \frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, a) = 0 .$$

Problem 1

$$\nabla^2 G(r,r';\theta) = \frac{1}{4\pi r^2}\delta(r-r')\delta(\Omega-\Omega')$$

$$G(r,r';\theta) = \sum_\ell G_\ell(r,r')\,P_\ell(\cos\theta)$$

$$\delta(\Omega-\Omega') = \sum \frac{2\ell+1}{4\pi} P_\ell(\cos\theta)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial G_\ell}{\partial r}\right) - \ell(\ell+1)G_\ell = \frac{2\ell+1}{4\pi}\delta(r-r')$$

$$v_1 = \left(\frac{r}{a}\right)^\ell \quad , \quad v_2 = \left(\frac{r}{a}\right)^\ell - \left(\frac{a}{r}\right)^{\ell+1}$$

$$w = \begin{vmatrix} \left(\frac{r}{a}\right)^\ell & \left(\frac{r}{a}\right)^\ell - \left(\frac{a}{r}\right)^{\ell+1} \\ \left(\frac{r}{a}\right)^{\ell-1}\frac{\ell}{a} & \left(\frac{r}{a}\right)^{\ell-1}\frac{\ell}{a} + \left(\frac{a}{r}\right)^{\ell+2}\frac{\ell+1}{a} \end{vmatrix} = \left(\frac{a}{r}\right)^2\frac{\ell+1}{a} + \left(\frac{a}{r}\right)^2\frac{\ell}{a} = \frac{2\ell+1}{a}\left(\frac{a}{r}\right)^2$$

$$G_\ell(r,r') = -\frac{2\ell+1}{4\pi}\,\frac{1}{-r^2\,\frac{2\ell+1}{a}\left(\frac{a}{r}\right)^2}\begin{cases}\left(\frac{r}{a}\right)^\ell\left[\left(\frac{r'}{a}\right)^\ell - \left(\frac{a}{r'}\right)^{\ell+1}\right], & r<r' \\ r \leftrightarrow r', & r>r'\end{cases}$$

$$r<r' \quad = -\frac{1}{4\pi}\frac{r^\ell}{r'^{\ell+1}} + \frac{1}{4\pi}\frac{r'^\ell}{\left(\frac{a^2}{r}\right)^{\ell+1}}\frac{a}{r}$$

$$G(r,r';\theta) = -\frac{1}{4\pi|\vec r-\vec r'|} + \frac{1}{4\pi\left|\hat r\frac{a^2}{r}-\vec r'\right|}\frac{a}{r}$$

Problem 2

$T = T_0 + \Delta T$, T_0: solution of static Laplace equation

$\Delta T = R(r)\tau(t)$

$$\frac{\nabla^2 R}{R} = \frac{\alpha^{-1}\frac{\partial\tau}{\partial t}}{\tau} \leftarrow \tau \propto e^{-\lambda t}$$

$$\nabla^2 R + \alpha^{-1}\lambda R = 0, \quad \frac{1}{r^2}\left(r^2 R'\right)' + \alpha^{-1}\lambda R = 0 \leftarrow R = \frac{u}{r}$$

$$\frac{1}{r^2}\left(r^2\left(-\frac{u}{r^2} + \frac{u'}{r}\right)\right)' + \alpha^{-1}\lambda\frac{u}{r} = 0$$

$$\frac{1}{r^2}\left(-u + u'r\right)' + \alpha^{-1}\lambda\frac{u}{r} = 0, \quad u'' + \alpha^{-1}\lambda u = 0$$

$$u \propto \left\{\begin{matrix}\sin\alpha^{-1}\lambda r \\ \cos\alpha^{-1}\lambda r\end{matrix}\right\}, \quad R \propto \frac{\sin\alpha^{-1}\lambda r}{r} \text{ (must be finite at } r=0\text{)}$$

$$T = T_0 + \sum_n A_n \frac{\sin\left(\overbrace{\sqrt{\alpha^{-1}\lambda_n a^2}}^{\equiv x_n}\,\frac{r}{a}\right)}{r} e^{-\lambda_n t}$$

a) $r = a$: $T_0 = T_0 + \sum_n A_n \frac{\sin x_n}{a} e^{-\lambda_n t} \Rightarrow \sin x_n = 0,\ x_n = \pi n$

$$\lambda_n = \frac{(\pi n)^2\alpha}{a^2}$$

b) $t = 0$: $0 = T_0 + \sum_n \frac{A_n \sin \pi n \frac{r}{a}}{r} \Rightarrow \sum_n A_n \frac{\sin \pi n y}{y} = -T_0 a$, $y = \frac{r}{a}$

$$A_n \int_0^1 \sin \pi n y \sin \pi m y \, dy = -T_0 a \int_0^1 \sin m\pi y \, dy, \quad A_n = \frac{2 T_0 a (-1)^n}{\pi n}$$

Problem 2 (cont'd.)

$$T = T_0 \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin \pi n \frac{r}{a}}{\pi n \frac{r}{a}} e^{-\frac{\pi^2 \alpha}{a^2} n^2 t} \right)$$

$$t \gg \frac{a^2}{\alpha \pi^2} : \quad T \approx T_0 \left(1 - 2 \frac{\sin \frac{\pi r}{a}}{\pi \frac{r}{a}} e^{-\frac{\pi^2 \alpha}{a^2} t} \right)$$

Problem 3

$$G = \sum_{m,n=0}^{\infty} \left(\frac{2}{a} \right)^2 \frac{1}{\varepsilon_n} \frac{\cos \frac{\pi n y}{a} \cos \frac{\pi n y'}{a} \sin \frac{\pi m x}{a} \sin \frac{\pi m x'}{a}}{\lambda - \left(\frac{\pi}{a} \right)^2 (n^2 + m^2)}$$

$$\varepsilon_n = \begin{cases} 1, & n \neq 0 \\ 0, & n = 0 \end{cases}$$