Spring 1995

Problem 1 (25pts+5pts+10pts=40pts)

A point source of neutrons on the axis of a long square column of graphite a on a side emits Q neutrons per unit time. Calculate the flux of neutrons on the axis at the distance z from the source if the diffusion coefficient of the neutrons is D. Find the closed form limiting expressions for (a) $z \gg a$ and (b) $z \ll a$.

Problem 2 (20pts)

Find the retarded Green's function of the one-dimensional Schrödinger operator $i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$.

Problem 3 (25pts+5pts+10pts=40pts)

A boundary eigenvalue problem for the Bessel equation: Formulate the normal mode problem for a clamped drum head of unit radius as a Sturm-Liouville eigenvalue problem, $L_n u = \lambda \rho u$, and find the eigenfunctions in terms of the zeros x_{nm} of the Bessel functions J_n. *Inhomogeneous boundary problem for the Bessel equation*: Find the Green's function of the operator L_n for (a) $n \neq 0$ and (b) $n = 0$.

MathPhys Final
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$$-D\nabla^2 n = Q\delta(\vec r-\vec r_0), \quad \nabla^2 n = -\frac{Q}{D}\delta(\vec r-\vec r_0), \quad r_0 = \left\{\frac{a}{2}, \frac{a}{2}, 0\right\}$$

$$\nabla^2 G = \delta(x-x')\delta(y-y')\delta(z-z')$$

$$\delta(x-x') = \frac{2}{a}\sum_n \sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}$$

$$\delta(y-y') = \frac{2}{a}\sum_m \sin\frac{m\pi y}{a}\sin\frac{m\pi y'}{a}$$

$$G = \left(\frac{2}{a}\right)^2 \sum_{mn} g_{nm}(z)\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{a}\sin\frac{m\pi y'}{a}$$

$$\left[-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{a}\right)^2\right]g_{mn} + \frac{d^2}{dz^2}g_{mn} = \delta(z-z')$$

$$(n^2+m^2)\left(\frac{\pi}{a}\right)^2 \equiv k_{nm}^2$$

$$\left(\frac{d^2}{dz^2}-k_{nm}^2\right)g_{nm} = \delta(z)$$

$$g_{nm}(z) = \int g_{nm}(\lambda)e^{i\lambda z}\,d\lambda$$

$$\delta(z) = \frac{1}{2\pi}\int e^{i\lambda z}\,d\lambda$$

$$(k_{nm}^2+\lambda^2)g_{nm}(\lambda) = -\frac{1}{2\pi}$$

$$g_{nm}(\lambda) = -\frac{1}{2\pi}\frac{1}{\lambda^2+k_{nm}^2}$$

$$g_{nm}(z) = -\frac{1}{2\pi}\int\frac{e^{i\lambda z}}{\lambda^2+k_{nm}^2}\,d\lambda = \begin{cases} -\dfrac{e^{-k_{nm}z}}{2k_{nm}}, & z> \\ -\dfrac{e^{k_{nm}z}}{2k_{nm}}, & z< \end{cases}$$

$$= -\frac{e^{-k_{nm}|z|}}{2k_{nm}}$$

$$G = -\frac{1}{2}\left(\frac{2}{a}\right)^2 \sum_{n,m} \frac{e^{-k_{nm}|z-z'|}}{k_{nm}}$$

$$\times \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y}{a} \sin\frac{m\pi y'}{a}$$

$$n = \frac{2Q}{Da^2} \sum_{n,m} \frac{e^{-k_{nm}|z|}}{k_{nm}} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \quad \sin\frac{n\pi}{2} \sin\frac{m\pi}{2}$$

$$= \frac{2Q}{Da^2} \sum_{n,m:\,odd} \frac{e^{-k_{nm}|z|}}{k_{nm}} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \, (-1)^{n+m} \to 1$$

$$j_z\Big|_{x=y=\frac{a}{2}} = -D\frac{\partial n}{\partial z} = \frac{2Q}{a^2} \sum_{n,m:\,odd} e^{-k_{nm}|z|}$$

a) $z \gg a$, $\quad j_z\left(\frac{a}{2}, \frac{a}{2}, z\right) \simeq \frac{2Q}{a^2} e^{-k_{11}|z|}$

b) $z \ll a$ $\quad \frac{n\pi}{a} = u, \quad \frac{m\pi}{a} = v, \quad \text{"}dn\text{"} = \frac{a\,du}{\pi}, \quad \text{"}dm\text{"} = \frac{a\,dv}{\pi}$

because summation over odd numbers

$$j_z\left(\frac{a}{2}, \frac{a}{2}, z\right) \simeq \frac{1}{4}\left(\frac{a}{\pi}\right)^2 \frac{2Q}{a^2} \iint_{0}^{\infty} du\,dv\, e^{-(u^2+v^2)^{1/2}|z|}$$

$$= \left(\frac{a}{\pi}\right)^2 \frac{2Q}{a^2} \frac{2\pi}{4} \frac{1}{4} \int_0^\infty dr\, r\, e^{-|z|r}$$

integration inside first quadrant

$$= \frac{Q}{4\pi|z|^2}$$

When $z \ll a \to$ spherical symmetry $\to j_z 4\pi z^2 = Q$

surface area

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)G(x,t)=\delta(x)\delta(t)$$

$$G=\int G(k,t)\,e^{ikx}\,dk$$

$$\delta(x)=\frac{1}{2\pi}\int e^{ikx}\,dk$$

$$\left(i\hbar\frac{d}{dt}-\frac{\hbar^2k^2}{2m}\right)G(k,t)=\frac{1}{2\pi}\delta(t)$$

$$\left(\frac{d}{dt}+\frac{i\hbar k^2}{2m}\right)G=\frac{1}{2\pi i\hbar}\delta(t)$$

$$G(k,t)=\frac{\theta(t)}{2\pi i\hbar}\,e^{-i\hbar k^2t/2m}$$

$$G(x,t)=\frac{\theta(t)}{2\pi i\hbar}\int dk\,e^{-\frac{i\hbar k^2t}{2m}+ikx}$$

$$=\frac{\theta(t)}{2\pi i\hbar}\int dk\,e^{-\frac{i\hbar t}{2m}\left(k-\frac{m}{\hbar t}x\right)^2}e^{\frac{imx^2}{2\hbar t}}$$

$$=\frac{\theta(t)}{2\pi i\hbar}\,e^{\frac{imx^2}{2\hbar t}}\sqrt{\frac{2m\pi}{i\hbar t}}$$

$$=-\theta(t)\sqrt{\frac{mi}{2\pi\hbar t}}\,e^{\frac{imx^2}{2\hbar t}}$$

$$(\nabla^2+k^2)\,\psi(r,\theta)=0, \qquad \psi(1,\theta)=0$$

$$\psi=R(r)\,\Theta(\theta), \quad \Theta(\theta)\propto e^{\pm in\theta}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(k^2-\frac{n^2}{r^2}\right)R=0, \qquad R(1)=0$$

$R(0)$: finit

$$R\propto J_n(kr), \quad k=x_{mn}$$

$$R\equiv u, \quad r\equiv x, \quad k^2\equiv\lambda$$

$$-(xu')'+\frac{n^2}{x}u\equiv L_n u=\lambda x u \qquad u(1)=0$$

$u(0)$: fini

$$p=x, \quad q=\frac{n^2}{x}, \quad \rho=x$$

$$L_n G(x,x')=\delta(x-x')$$

$$L_n v=-(xv')'+\frac{n^2}{x}v=0$$

a) $n\neq 0 \qquad v''+\frac{1}{x}v'-\frac{n^2}{x^2}v=0 \longleftarrow v\propto x^\beta$

$$\beta(\beta-1)+\beta-n^2=0, \quad \beta=\pm n, \quad v\propto x^{\pm n}$$

$$v_1(x)=x^n, \quad v_2(x)=x^n-x^{-n}; \quad \Big\}\ v_2(1)=0$$

$$W=\begin{vmatrix} x^n & x^n-x^{-n} \\ nx^{n-1} & n(x^{n-1}+x^{-n-1}) \end{vmatrix}=\frac{2n}{x}$$

Problem 3 (continued)

$$G(x,x') = -\frac{1}{24}\begin{cases} x^4\left(x'^4 - x'^{-4}\right), & 0 \le x \le x' \le 1 \\ x'^4\left(x^4 - x^{-4}\right), & 0 \le x' \le x \le 1 \end{cases}$$

b) $n = 0$ $\quad (xV')' = 0$

$v_1(x) = 1, \quad v_2(x) = \ln x; \quad \{ v_2(1) = 0$

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$$

$$G = -\begin{cases} \ln x', & 0 \le x \le x' \le 1 \\ \ln x, & 0 \le x' \le x \le 1 \end{cases}$$