

Bessel function

①

A. $x \rightarrow 0 \quad x^2 y'' + x y' - m^2 y = 0, \quad y = x^s$

$$s(s-1) + s - m^2 = 0, \quad s = \pm m$$

B. $y = x^{\pm m} u$

$$x u'' + (1 \pm 2m) u' + x u = 0,$$

$$u = \sum_{n=0}^{\infty} C_n x^n$$

$$[n(n+2) + (1 \pm 2m)] C_{n+2} + C_n = 0$$

$$[(n+2 \pm 2m)(n+2)] C_{n+2} = -C_n, \quad C_{n+2} = C_n \frac{-1}{(n+2)(n+2 \pm 2m)}$$

$$\sum_{n=0}^{\infty} n(n-1) C_n x^{n-1} + \sum_{n=0}^{\infty} (1 \pm 2m) n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n \pm 2m) C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n \pm 2m) C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+1} + C_1 (1 \pm 2m) = 0$$

$$n-2 \rightarrow n$$

$$\div C_1 (1 \pm 2m) = 0$$

$$C_1 = 0, \text{ or } m = \mp \frac{1}{2}$$

a) $m = -\frac{1}{2}$, then $C_1 (1+2m) = 0$ leads to $C_1 = 0$ (and $C_{2k+1} = 0$)

for $y_+ = x^{1/2} \sum_{n=0}^{\infty} C_n^{(+)} x^n$, but $C_n^{(+)} = C_{n+1}^{(-)}$

analogously

b) $m = \frac{1}{2}$

for $y_- = x^{-1/2} \sum_{n=0}^{\infty} C_n^{(-)} x^n$ part of y_- with $C_{2k+1}^{(-)}$ coincides with $C_{2k}^{(+)}$

C. m: integer > 0 (≥ 1) *

$$C_n^{(-)} = -C_{n+2}^{(-)} (n+2)(n+2-2m)$$

$$C_n^{(-)} \leq 2m-2 = 0$$

$$y_- = x^{-m} \sum_{\substack{n \geq 2m \\ n: \text{even}}} C_n^{(-)} x^n$$

$$y_- = x^m \sum_{\substack{n-2m \geq 0 \\ n: \text{even}}} C_n^{(-)} x^{n-2m} = x^m \sum_{\substack{k \geq 0 \\ k: \text{even}}} C_{k+2m}^{(-)} x^k$$

$$y_+ = x^m \sum_{\substack{k \geq 0 \\ k: \text{even}}} C_k^{(+)} x^k$$

$$\frac{C_{k+2+2m}^{(-)}}{C_{k+2m}^{(-)}} = \frac{-1}{(k+2+2m)(k+2+2m-2m)} = \frac{1}{(k+2)(k+2+2m)} = \frac{C_{k+2}^{(+)}}{C_{k+2}^{(+)}}$$

$$\frac{y_-}{y_+} = \text{const}$$

* m=0

$$Ly(x,s) = s^2 x^s$$

$$\frac{\partial}{\partial s} Ly(x,s) \Big|_{s=0} = 2sx^s + s^2 \ln x x^s \Big|_{s=0} = 0$$

$$\bar{y} = \frac{\partial}{\partial s} y(x,s) \Big|_{s=0} \text{ : second solution } \leftarrow L \left(\frac{\partial}{\partial s} y(x,s) \Big|_{s=0} \right) = 0$$

Declarations

$$\square x^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} y + x \frac{\partial}{\partial x} y + (x^2 - m^2) y = 0$$

$$\square y = u x^m$$

$$\Delta (x^2 - m^2) u x^m + \left(2m x^{m-1} \frac{\partial}{\partial x} u + x^m \frac{\partial}{\partial x} \frac{\partial}{\partial x} u + [m-1] m u x^{m-2} \right) x^2 + \left(x^m \frac{\partial}{\partial x} u + m u x^{m-1} \right) x = 0 \quad \text{Substitute}$$

$$\Delta x^{m+1} \frac{\partial}{\partial x} u + 2m x^m + 1 \frac{\partial}{\partial x} u + x^{m+2} \frac{\partial}{\partial x} \frac{\partial}{\partial x} u + u x^{m+2} = 0 \quad \text{Expand}$$

$$x^{m+1} \frac{\partial}{\partial x} u + 2m x^m + 1 \frac{\partial}{\partial x} u + x^{m+2} \frac{\partial}{\partial x} \frac{\partial}{\partial x} u + u x^{m+2}$$

$$\Delta \frac{x^{m+1}}{x^{m+1}} = \frac{0}{x^{m+1}} \quad \text{Apply}$$

$$\Delta \frac{\partial}{\partial x} u + 2m \frac{\partial}{\partial x} u + x \frac{\partial}{\partial x} \frac{\partial}{\partial x} u + u x = 0 \quad \text{Expand}$$

$$\Delta (2m+1) \frac{\partial}{\partial x} u + x \frac{\partial}{\partial x} \frac{\partial}{\partial x} u + u x = 0 \quad \text{Collect}$$

$$\square y = x^{-m} v$$

$$\Delta \frac{(x^2 - m^2)v}{x^m} + \left(-2m x^{-m-1} \frac{\partial}{\partial x} v + \frac{1}{x^m} \frac{\partial}{\partial x} \frac{\partial}{\partial x} v - [-m-1] m v x^{-m-2} \right) x^2 + \left(\frac{1}{x^m} \frac{\partial}{\partial x} v - m v x^{-m-1} \right) x = 0 \quad \text{Substitute}$$

$$\Delta x^{-m+1} \frac{\partial}{\partial x} v - 2m x^{-m+1} \frac{\partial}{\partial x} v + x^{-m+2} \frac{\partial}{\partial x} \frac{\partial}{\partial x} v + v x^{-m+2} = 0 \quad \text{Expand}$$

$$\Delta \left(x^{-m+1} \frac{\partial}{\partial x} v - 2m x^{-m+1} \frac{\partial}{\partial x} v + x^{-m+2} \frac{\partial}{\partial x} \frac{\partial}{\partial x} v + v x^{-m+2} \right) x^{m-1} = 0 x^{m-1} \quad \text{Apply}$$

$$\Delta \frac{\partial}{\partial x} v - 2m \frac{\partial}{\partial x} v + x \frac{\partial}{\partial x} \frac{\partial}{\partial x} v + v x = 0 \quad \text{Expand}$$

$$\Delta (-2m+1) \frac{\partial}{\partial x} v + x \frac{\partial}{\partial x} \frac{\partial}{\partial x} v + v x = 0 \quad \text{Collect}$$