Fourier and Laplace Transforms

Math Physics
Quiz 3

11-15-2007

1. In the presence of friction, the equation of motion of a harmonic oscillator is given by

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

The oscillator, initially at rest, is driven by a constant force $F(t) = F_0\theta(t)$, where $\theta(t)$ is the step function. Determine the motion of the oscillator using any two of the following three techniques:

(a) Fourier transform,

(b) Laplace transform,

(c) Solving the equation directly.

Assume that $\omega_0 > \lambda$. Verify your answer for the two limits: frictionless motion, $\lambda = 0$, and the long-time behavior, $t \gg \lambda^{-1}$.

Important: you can get up to 85% of the total points by solving just the frictionless limit, $\lambda = 0$.

Solution

(a) Fourier transform

$$X = -\frac{iF_0}{2\pi m\omega\left(-\omega^2 + 2i\lambda\omega + \omega_0^2\right)}$$

where

$$\theta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{i\omega} d\omega$$

was used. Then

$$\begin{aligned} x(t) &= -\int \frac{iF_0 \exp(i\omega t)\, d\omega}{2\pi m\omega\left(-\omega^2 + 2i\lambda\omega + \omega_0^2\right)} \\ &= \oint \frac{iF_0 \exp(i\omega t)\, d\omega}{2\pi m\omega\left(\omega - \omega_1\right)\left(\omega - \omega_2\right)} \end{aligned}$$

where the contour is closed in the upper plane, the pole at $\omega = 0$ is shifted upwards, to be inside the contour, and

$$\begin{aligned} \omega_{1,2} &= \pm\left(\omega_0^2 - \lambda^2\right)^{1/2} + i\lambda \\ \omega_1\omega_2 &= -\omega_0^2 \end{aligned}$$

Evaluating the integral by residues, find

$$\begin{aligned} x(t) &= -\frac{F_0}{m}\left[\frac{1}{\omega_1\omega_2} + \frac{1}{(\omega_1 - \omega_2)}\left(\frac{\exp(i\omega_1 t)}{\omega_1} - \frac{\exp(i\omega_2 t)}{\omega_2}\right)\right] \\ &= \frac{F_0}{m\omega_0^2}\left[1 - \frac{\omega_1 \exp(i\omega_2 t) - \omega_2 \exp(i\omega_1 t)}{\omega_1 - \omega_2}\right] \end{aligned}$$

Clearly, $x(0) = 0$ and $\dot{x}(0) = 0$. In the limit $\lambda = 0$,

$$x(t) = \frac{F_0}{m\omega_0^2}\left[1 - \cos(\omega_0 t)\right]$$

After time $t \gg \lambda^{-1}$, the transients vanish (because of the factor $\exp(-\lambda t)$) and

$$x(t) = \frac{F_0}{m\omega_0^2}$$

(b) Laplace transform

$$\left(s^2 + 2\lambda s + \omega_0^2\right) X = \frac{F_0}{ms}$$

whereof, by Laplace inversion formula,

$$x(t) = \frac{1}{2\pi i}\frac{F_0}{m}\int \frac{ds \exp st}{(s - s_1)(s - s_2)\, s}$$

where the contour is a vertical line to the right of zero and $s_{1,2}$ are given by

$$\begin{aligned} s_{1,2} &= -\lambda \pm i\left(\omega_0^2 - \lambda^2\right)^{1/2} \\ s_1 s_2 &= \omega_0^2 \end{aligned}$$

Closing contour to the left and evaluating the residues

$$\begin{aligned} x(t) &= \frac{F_0}{m}\left[\frac{1}{s_1 s_2} + \frac{\exp s_1 t}{s_1(s_1 - s_2)} + \frac{\exp s_1 t}{s_2(s_2 - s_1)}\right] \\ &= \frac{F_0}{m\omega_0^2}\left[1 + \frac{s_2 \exp(s_1 t) - s_1 \exp(s_2 t)}{s_1 - s_2}\right] \end{aligned}$$

(c) Solving the equation directly

Particular solution

$$x_p = \frac{F_0}{m\omega_0^2}$$

Homogeneous solution

$$x_h(t) \propto \exp \omega t$$

whereof

$$\omega_{1,2} = -\lambda \pm i\left(\omega_0^2 - \lambda^2\right)^{1/2}$$

and

$$x(t) = A \exp \omega_1 t + B \exp \omega_2 t + \frac{F_0}{m\omega_0^2}$$

From $x(0) = 0$ and $\dot{x}(0) = 0$,

$$A + B = -\frac{F_0}{m\omega_0^2},\ A\omega_1 + B\omega_2 = 0$$

Therefore

$$A = \frac{F_0}{m\omega_0^2}\frac{\omega_2}{\omega_1 - \omega_2},\ B = -\frac{F_0}{m\omega_0^2}\frac{\omega_1}{\omega_1 - \omega_2}$$

and

$$x(t) = \frac{F_0}{m\omega_0^2}\left[1 + \frac{\omega_2 \exp(\omega_1 t) - \omega_1 \exp(\omega_2 t)}{\omega_1 - \omega_2}\right]$$